Explicit examples of minimal hypersurfaces: a four-fold periodic minimal hypersurface in $\mathbb{R}^4$ and beyond

Vladimir Tkachev

Linköping University

Why explicit examples?

- Consider a homogeneous cubic form in $\mathbb{R}^5$

$$u_1(x) = x_5^3 + \frac{3}{2}x_5(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2) + \frac{3\sqrt{3}}{2}(x_4(x_2^2 - x_1^2) + 2x_1x_2x_3).$$

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$$u_d(x) := \frac{3\sqrt{3}}{2}\det\begin{pmatrix} x_2 - \frac{1}{\sqrt{3}}x_1 & \bar{z}_1 & \bar{z}_2 \\ z_1 & -x_2 - \frac{1}{\sqrt{3}}x_1 & \bar{z}_3 \\ z_2 & z_3 & \frac{2}{\sqrt{3}}x_1 \end{pmatrix}, \quad x \in \mathbb{R}^{3d+2}, \qquad (1)$$

are the only isoparametric polynomials corresponding to hypersurfaces in the Euclidean spheres having exactly 3 distinct constant principal curvatures. Here $z_k \in \mathbb{R}^d \cong \mathbb{F}_d$ is the real division algebra of dimension $d \in \{1, 2, 4, 8\}$.

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$$|Du(x)|^2 = 9|x|^4, \qquad \Delta u(x) = 0, \quad x \in \mathbb{R}^n.$$

- It can be shown that

$$u_d(x) = \sqrt{2}N(x), \qquad x \in J_0,$$

where J_0 is the trace free subspace of the formally real Jordan algebra $J = \mathfrak{h}_3(\mathbb{F}_d)$, $d = 1, 2, 4, 8$.

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- N. Nadirashvili, S. Vlăduţ, V.T. (2011): $u_1(x)|x|^{-1}$ is a viscosity solution to a uniformly elliptic Hessian equation $F(D^2u) = 0$ in the unit ball in $\mathbb{R}^5$.

Some general remarks

Searching of explicit examples: they are also ‘minimal’ in the sense that they are very distinguished in many respects.

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Searching of explicit examples: they are also 'minimal' in the sense that they are very distinguished in many respects.

- Classical (2D) minimal surface theory relies heavily on the Weierstrass-Enneper representation and complex analysis tools (uniqueness theorem, reflection principle etc)
- The codimension two case is also very distinguished: any complex hypersurface in $\mathbb{C}^n = \mathbb{R}^{2n}$ is always minimal.
- The only known explicit examples of complete minimal hypersurfaces in $\mathbb{R}^n$, $n \geq 3$, are the catenoids, and minimal hypercones (in particular, the isoparametric ones). There are also known to exist some minimal graphs in $\mathbb{R}^n$, $n \geq 9$ (E. Bombieri, de Giorgi, E. Giusti, L. Simon), the (immersed) analogues of Enneper's surface by J. Choe in $\mathbb{R}^n$ for $4 \leq n \leq 7$; the embedded analogues of Riemann one-periodic examples due to S. Kaabachi, F. Pacard in $\mathbb{R}^n$, $n \geq 3$. None of the latter examples are known explicitly.
- W.Y. Hsiang (1967): find an appropriate classification of minimal hypercones in $\mathbb{R}^n$, at least of cubic minimal cones.
- V.T. (2012): It turns out that the most natural framework for studying cubic minimal cones is *Jordan algebras* (non-associative structures frequently appeared in connection with elliptic type PDE's); will be discussed later.

Minimal surfaces with ‘harmonic level sets’

Minimal surfaces with 'harmonic level sets'

Let $h(z):\mathbb{C}^m\to\mathbb{C}$ be holomorphic and $F:\mathbb{R}^k\to\mathbb{R}$ be C^2-smooth real valued. Define a hypersurface by the implicit equation

$$M=\{x\in\mathbb{R}^{2m+k}=\mathbb{C}^m\times\mathbb{R}^k:\ F(\mathbf{t})=\operatorname{Re}h(\mathbf{z})\}$$

Then M is minimal at its regular points if and only if

$$\operatorname{Re}\sum_{\alpha,\beta=1}^{m}h''_{\alpha\beta}\bar{h}'_{\alpha}\bar{h}'_{\beta}+|\nabla h(z)|^2\Delta F(t)-\Delta_1(F)\equiv 0 \mod (F(\mathbf{t})-\operatorname{Re}h(\mathbf{z}))$$

where

$$\Delta_1F=|\nabla F|^2\Delta F-\sum_{i,j=1}^{k}F''_{ij}F'_iF'_j$$

is the mean curvature operator.

Remark. Some important particular cases: $F\equiv 0$, $h\equiv 0$. For instance, the Lawson minimal cones in $\mathbb{R}^{2n}$ produced by $h(z_1,\ldots,z_n)=z_1^{m_1}\ldots z_n^{m_n}$, $F\equiv 0$. In fact, there are many other examples.

Minimal surfaces with 'harmonic level sets'

Consider the case $k = m = 1$. Then

$$M = \{x \in \mathbb{R}^3 : F(x_1) = \operatorname{Re} h(x_1 + x_2\sqrt{-1})\},$$

can be thought of as a hypersurface $\mathbb{R}^3$ with 'harmonic level sets'.

V.V. Sergienko, V.T. (1998)

The surface M above is minimal if and only if $h'(z) = 1/g(z)$ ($z = x_1 + x_2\sqrt{-1}$) with $g(z)$ satisfying

$$g''(z)g(z) - g'^2(z) = c \in \mathbb{R}.$$

Then the function F is found by $F''(t) + Y(F(t)) = 0$, where $Y(t)$ is well-defined by virtue of

$$\frac{\operatorname{Re} g'}{|g|^2} = -Y(\operatorname{Re} h(z)).$$

Remark. Notice that the resulting surface M, if non-empty, is automatically embedded because

$$|\nabla u(x)|^2 = F'^2(x_1) + \frac{1}{|g(z)|^2} > 0,$$

where $u(x) = F(x_1) - \operatorname{Re} h(x_1 + x_2\sqrt{-1})$.

Minimal surfaces with ‘harmonic level sets’

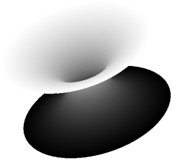
The only possible solutions of $g''(z)g(z) - g'^2(z) = c \in \mathbb{R}$ are

(a) $g(z) = az + b$,

(b) $g(z) = ae^{bz}$, and

(c) $g(z) = a\sin(bz + c)$, $a^2, b^2 \in \mathbb{R}$ and $c \in \mathbb{C}$

which, in particular, yields:

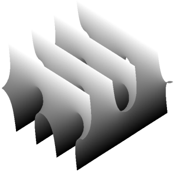


the catenoid

$g(z) = z$
$h(z) = \ln z$

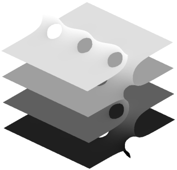
the helicoid

$g(z) = \mathrm{i}z$
$h(z) = -\mathrm{i}\ln z$



a Scherk type surface

$g(z) = e^z$
$h(z) = -e^{-z}$



a doubly periodic surface

$g(z) = \sin z$
$h(w) = -\ln\tanh\frac{z}{2}$

The defining equations

$x_1^2 + x_2^2 = \cosh^2 x_3$

$\frac{x_2}{x_1} = \tan x_3$

$\exp x_3 = \frac{\cos x_2}{\cos x_1}$

$\mathrm{cn}(\frac{kx_3}{k'}, k) = \frac{\sin x_2}{\sinh x_1}$

Triply-periodic minimal surfaces in $\mathbb{R}^3$

Observe that all the above solutions have the following multiplicative form:

$$\phi_1(x_1)\phi_2(x_2)\phi_3(x_3) = 1. \quad (2)$$

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V. Sergienko and V.T. (1998)

Let (2) define a minimal surface in $\mathbb{R}^3$. Suppose also that two of the functions ϕ_i assume a zero value and satisfy $\phi_k'^2 = P_k(\phi_k^2)$ for all $k = 1, 2, 3$. Then

$$\begin{aligned} {\phi_1'}^2 &= a_{11} + a_{12}\phi_1^2 + a_{13}\phi_1^4 \\ {\phi_2'}^2 &= a_{21} + a_{22}\phi_2^2 + a_{23}\phi_2^4 \\ {\phi_3'}^2 &= a_{31} + a_{32}\phi_3^2 + a_{33}\phi_3^4 \end{aligned}$$

where the matrix

$$A' := \begin{pmatrix} a_{11} & \frac{1}{2}(a_{22} + a_{23}) & a_{13} \\ a_{21} & \frac{1}{2}(a_{21} + a_{23}) & a_{23} \\ a_{31} & \frac{1}{2}(a_{21} + a_{22}) & a_{33} \end{pmatrix}$$

is *generating*, i.e. $a'_{i\alpha}a'_{i\beta} = a'_{j\gamma}a'_{k\gamma}$.

Remark. The corresponding results hold also true for maximal surfaces in the 3D Minkowski space-time (Sergienko V.V., V.T.; see some related results due to J. Hoppe).

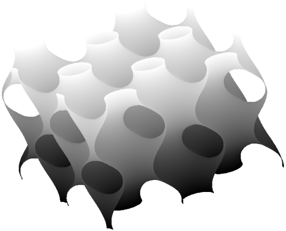
Triply-periodic minimal surfaces in $\mathbb{R}^3$

Example.

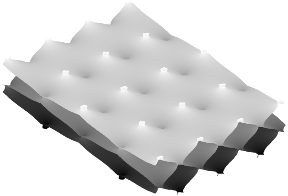
$$A' = \begin{pmatrix} \frac{1}{1+k^2} & -\frac{(1+k^2)m^2}{(1+m^2)(1-k^2m^2)} & -\frac{k^2}{1+k^2} \\ \frac{1}{1+m^2} & -\frac{(1+m^2)k^2}{(1+k^2)(1-k^2m^2)} & -\frac{m^2}{1+m^2} \\ \frac{k^2m^2}{1-k^2m^2} & \frac{1-k^2m^2}{(1+k^2)(1+m^2)} & \frac{1}{1-k^2m^2} \end{pmatrix}, \qquad k^2m^2 \leq 1,$$

the corresponding triply periodic minimal surface

$$km\ \mathrm{sn}(\frac{x_3}{\sqrt{1-k^2m^2}};\ km) = \mathrm{cn}(x_1;\ \frac{1}{\sqrt{1+k^2}})\ \mathrm{cn}(x_2;\ \frac{1}{\sqrt{1+m^2}})$$



A triply periodic minimal surface with $k = m = \frac{1}{\sqrt{2}}$



A porous gasket ($k = 8$ and $m = \frac{1}{9}$)

Some further motivations and observations

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Some known examples of minimal hypersurfaces in $\mathbb{R}^4$ in the *additive form*

$$\phi_1(x_1) + \phi_2(x_2) + \phi_3(x_3) + \phi_4(x_4) = 0$$

- a hyperplane, $a_1 y_1 + a_2 y_2 + a_3 y_3 + a_4 y_4 = 0$
- the Clifford cone, eq. I: $\ln x_1 + \ln x_2 - \ln x_3 - \ln x_4 = 0$ (actually, $x_1 x_2 = x_3 x_4$)
- the Clifford cone, eq. II: $x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0$;
- the 3D-catenoid: $x_1^2 + x_2^2 + x_3^2 - \frac{1}{\mathrm{sn}^2(x_4, \sqrt{-1})} = 0$

Remarks.

(i) The 3D-catenoid can be thought as a one-periodic minimal hypersurface in $\mathbb{R}^4$ (cf. the Riemann minimal surface in $\mathbb{R}^4$).

(ii) The same elliptic function $\mathrm{sn}(x, \sqrt{-1})$ also appears in the four-fold periodic example considered below.

Some further motivations and observations

A basic regularity property of n-dimensional *area minimizing (or stable minimal)* hypersurfaces is that the Hausdorff dimension of the singular set is less than or equal to $n-7$ (F. Almgren, E. Giusti, J. Simons, L. Simon, R. Schoen).

Important questions:

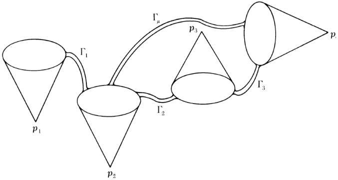
- What kinds of singular sets actually occur?
- In particular, it would be interesting to characterize embedded minimal submanifolds of $\mathbb{R}^n$ which have isolated singularities.

Remark. On the other hand, isolated singularities for solutions of the *maximal surface equation* in $\mathbb{R}^n$, $n \geq 2$, is a very common phenomenon (E. Calabi, L. Simon, R. Bartnik, K. Ecker, V. Miklyukov, A.A. Klyachin, V.A. Klyachin, I. Fernández, F.J. López, R. Souam). Here, the most natural problem is to characterize the singular data (the singular set with a prescribed asymptotic behaviour) which guarantee the existence of the corresponding maximal surfaces.

Some other motivations and observations

Minimal hypersurfaces in $\mathbb{R}^n$ with singularities:

- L. Caffarelli, R. Hardt, L. Simon (1984) showed that there exist (bordered) embedded minimal hypersurfaces in $\mathbb{R}^n$, $n \geq 4$, with an isolated singularity but which is not a cone.
- N. Smale (1989) proved the existence of examples of stable embedded minimal hypersurfaces with boundary, in $\mathbb{R}^n$, $n \geq 8$, with an arbitrary number of isolated singularities.



Source: *Annals of Mathematics*, Second Series, Vol. 130, No. 3 (Nov., 1989), pp. 603-642

A four-fold periodic minimal hypersurface

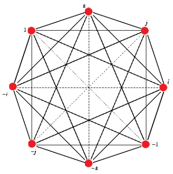
Three remarkable lattices in $\mathbb{R}^4 \cong \mathbb{H}$

We identify a vector $x \in \mathbb{R}^4$ with the quaternion $x_1\mathbf{1} + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4 \in \mathbb{H}$.

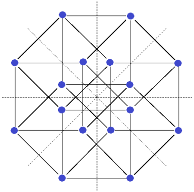
- the *checkerboard lattice*: $D_4 = \{m \in \mathbb{Z}^4 : \sum_{i=1}^4 m_i \equiv 0 \mod 2\}$
- the *Lipschitz integers*: $\mathbb{Z}^4 = \{m \in H : m_i \in \mathbb{Z}\} = D_4 \sqcup (\mathbf{1} + D_4)$
- the F_4 lattice of the *Hurwitz integers* $\mathcal{H} = \mathbb{Z}^4 \sqcup (\mathbf{h} + \mathbb{Z}^4)$, where

$$\mathbf{h} = \frac{1}{2}(\mathbf{1} + \mathbf{i} + \mathbf{j} + \mathbf{k}),$$

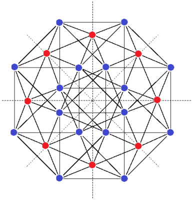
  is an abelian ring (the densest possible lattice packing of balls in $\mathbb{R}^4$).



$\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$
the 16-cell



$\frac{1}{2}(\pm\mathbf{1} \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})$
the 8-cell (the hypercube)



...taken together
the 24-cell

(after John C. Baez, *Bull. Amer. Math. Soc.*, 2002)

Lemniscatic functions

Define the *lemniscatic sine* by

$$\mathrm{sl}'^2\, t = 1 - \mathrm{sl}^4\, t, \quad \mathrm{sl}\, 0 = 0, \tag{3}$$

or by Jacobi's elliptic sine function: $\mathrm{sl}\, t = \mathrm{sn}(x, \sqrt{-1})$, and denote by

$$s(t) := \mathrm{sl}(\varpi t),$$

and the associate lemniscatic cosine function

$$c(t) = s(t + \frac{1}{2}),$$

where

$$\varpi = 2\omega = \int_{-1}^{1} \frac{dt}{\sqrt{1-t^4}} = \frac{\Gamma(\frac{1}{4})^2}{2\sqrt{2\pi}}.$$

Some important properties:

- the double periodicity: $s(z + 2n_1 + 2\mathbf{i}\, n_2) = s(z)$, $n_1, n_2 \in \mathbb{Z}$;
- the multiplicative identity: $s(\sqrt{-1}z) = \sqrt{-1}s(z)$.
- The Euler-Fangano identity:

$$(1 + s^2(t))(1 + c^2(t)) = 2.$$

The construction

Define $S(x) = s(x_1)s(x_2) - s(x_3)s(x_4) : \mathbb{R}^4 \to \mathbb{R}$ and

$$M := S^{-1}(0)$$

and define the skeleton of M by $M_0 := \{x \in \mathbb{R}^4 : s(x_1)s(x_2) = s(x_3)s(x_4) = 0\}$.

Proposition 1

(i) $M \setminus \mathrm{Sing}(M)$ is a smooth embedded minimal hypersurface in $\mathbb{R}^4$, where

$$\mathrm{Sing}(M) = \mathbb{Z}^4 \sqcup (\mathbf{h} + D_4),$$

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Proof. It is straightforward to verify that

$$\Delta_1 S(x) = S(x) \cdot \text{a polynomial of } s_i$$

where Δ_1 is the mean curvature operator. Furthermore,

$$\frac{1}{\varpi^2}|\nabla S|^2 \equiv (s_1^2 + s_2^2 + s_3^2 + s_4^2)(1 - s_1^2 s_2^2) \mod S.$$

Thus, $|\nabla S|$ vanishes at $x \in M$ if either of the following holds:

- $s(x_1) = s(x_2) = s(x_3) = s(x_4) = 0 \Leftrightarrow x \in \mathbb{Z}^4$ (singularities of $\mathbb{Z}^4$-type)
- $s(x_1)^2 s(x_2)^2 = 1 \Leftrightarrow x \in \mathbf{h} + D_4$ (singularities of D_4-type)

The singular set of M

- Singularities of $\mathbb{Z}^4$-type: if $a \in \mathbb{Z}^4$ then
$$S(a+x) = \pm x_1 x_2 \pm x_3 x_4 + O(|x|^4), \qquad \text{as } x \to 0.$$
- Singularities of D_4-type: if $a =\in \mathbf{h} + D_4$ then
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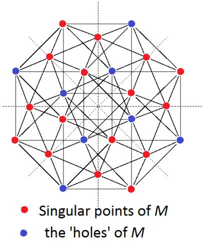
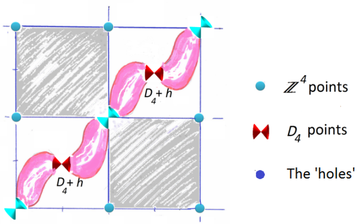
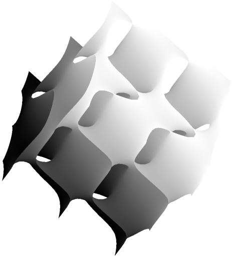
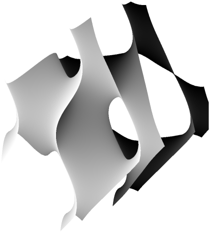


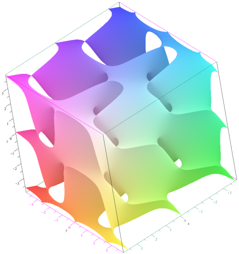


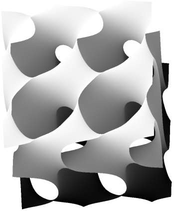
Question: If there exists a symmetry of $\mathbb{R}^4$ which interchanges the two types?

Some cross-section chips of M









A stratification of M

A stratification of M

The function $\sigma(x) = \frac{2}{\pi}\arcsin(s(x_1)s(x_2)) : M \to \mathbb{R}$ is well-defined smooth function on $M \setminus \mathbb{Z}^4$ with

$$K \cdot |\nabla_M \sigma|^2 = \frac{(s_1^2+s_2^2)\cdot(s_3^2+s_4^2)}{s_1^2+s_2^2+s_3^2+s_4^2}, \qquad s_i = s(x_i),\ K \in \mathbb{R}.$$

The level sets $\sigma^{-1}(\lambda)$, $-1 \le \lambda \le 1$, foliate M as follows:

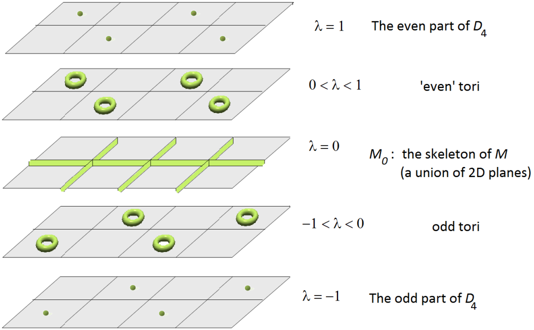


Figure. A singular foliation of M by the level sets of σ

The connectedness of M

Proposition 2

M is path-connected.

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Proof. We will show that any point $x \in M$ can be connected wit the origin in $\mathbb{R}^4$. Notice that $\sigma(A) = \sigma(B) =: \lambda$, where $\sigma(y) = s(y_1)s(y_2) : \mathbb{R}^2 \to \mathbb{R}$ and $A = (x_1, x_2)$, $A' = (x_3, x_4)$, and observe that $x = (y, y') \in M$ if and only if $\sigma(y) = \sigma(y')$.

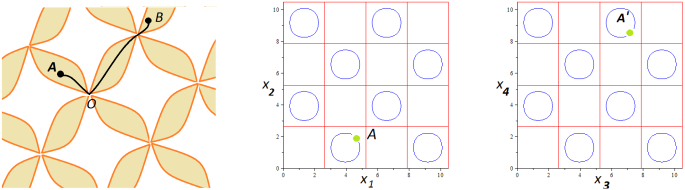

Figure: The λ-level set of σ in the (x_1, x_2)- and (x_3, x_4)-planes respectively.

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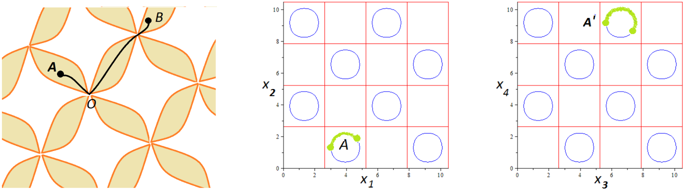

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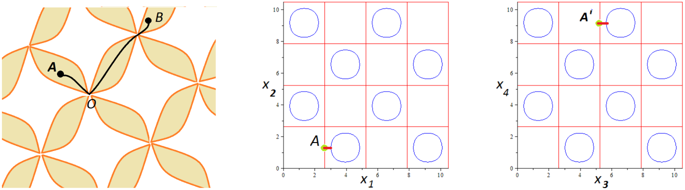

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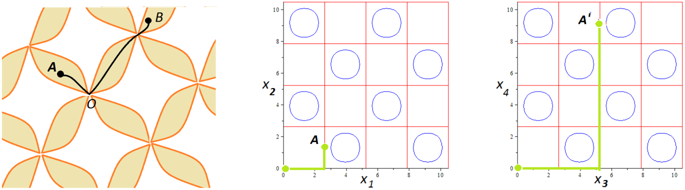

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Symmetries of M

Proposition 3

Let $T(M)$ be the group the translations of $\mathbb{R}^4$ leaving M invariant. Then

$$T(M) = D_4.$$

Proof. The inclusion $D_4 \subset T(M)$ is by the anti-periodicity $s(x+1) = -s(x)$.

In the converse direction, suppose $x \to m + x$ is in $T(M)$.

- Then $S(x_1, 0, x_3, 0) = 0$ implies $(x_1, 0, x_3, 0) \subset M$, and, thus, also $(x_1 + m_1, m_2, x_3 + m_3, m_4) \subset M$, i.e.

$$s(x_1 + m_1)s(m_2) = s(x_3 + m_2)s(m_4) \qquad \text{for } \textit{any} \quad x_1, x_3 \in \mathbb{R}.$$

  This implies $m_1, m_3 \in \mathbb{Z}$. Similarly, $m_2, m_4 \in \mathbb{Z}$.
  Thus $m \in \mathbb{Z}^4$.
- On the other hand, $\mathbf{h} \in M$ and $s(m_i + \frac{1}{2}) = (-1)^{m_i}$ imply

$$(-1)^{m_1+m_2} = (-1)^{m_3+m_4}.$$

  Thus, $m \in D_4$, as required. □

Symmetries of M

Symmetries of M

Proposition 4

The group of the orthogonal transformations of $\mathbb{R}^4$ leaving M invariant is isomorphic to $\mathcal{D}_4 \times \mathbb{Z}_2^4$, where $\mathcal{D}_4$ is the dihedral group (the symmetry group of the square).

Proof. Let $O(M)$ be the group of orthogonal automorphisms of M and define

$$\Sigma_0 = \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$$

$$\Sigma_1 = \{(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}) : \text{ an even number minus signs}\}$$

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Each set consists of 8 unit vectors (the so-called Hurwitz units).

Suppose $U \in O(M)$. Then one can show that:

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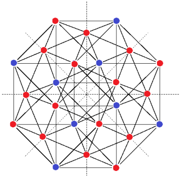

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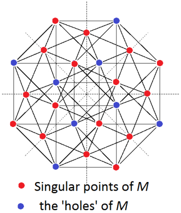

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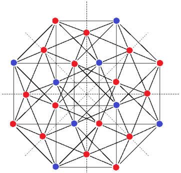

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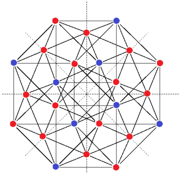

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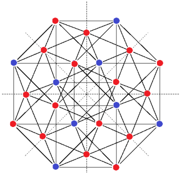

- Singular points of M
- the 'holes' of M

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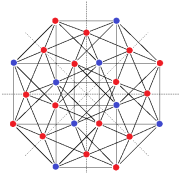

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- It follows that U acts also as a permutation on the set $\{\Sigma_i : i = 1..3\}$ stabilizing Σ_2.

Symmetries of M

Next, since $U : M \to M$, one easily finds that the quadratic form

$$q(x) = x_1x_2 - x_3x_4$$

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On the other hand, using the fact that $U : \Sigma_2 \to \Sigma_2$ and choosing an orthonormal basis in Σ_2, denoted by v_i, $i = 1, 2, 3, 4$, as the column-vectors of the matrix

$$B := \frac{1}{2} \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

we find for any $x = \sum_{i=1}^{4} y_i v_i$

$$q(x) = y_3^2 + y_4^2 - y_1^2 - y_2^2$$

and we also have

$$U v_i = \epsilon_i v_{\alpha(i)}, \qquad 1 \leq i \leq 4, \quad \epsilon_i^2 = 1,$$

where α is a permutation which is actually is an element of the symmetry group of the square (the dihedral group) $\mathcal{D}_4$.

It follows that $O(M)$ is a subgroup of the group of orthogonal transformations leaving Σ_2 invariant, i.e. $\subset \mathcal{D}_4 \times \mathbb{Z}_2^4$.

Proposition 5

Let

$$\Phi(x) := \frac{s(x_1)s(x_2) - s(x_3)s(x_4)}{(1 + s^2(\frac{x_1+x_2}{2})s^2(\frac{x_3+x_4}{2})) \cdot (1 + s^2(\frac{x_1-x_2}{2})s^2(\frac{x_3-x_4}{2}))}$$

and

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Then

- $\Phi^{-1}(0) = M$
- $\Phi(Ax) = \Phi(x)$,
- A is a reflection in $\mathbb{R}^4$ leaving invariant the 'holes' $D_4 + \mathbf{h}^2$ and $A : \Sigma_2 \to \Sigma_2$. More precisely, A is an orthogonal involution Σ_0 onto Σ_1;

It follows, in particular, that M is invariant under the A-action.

Some remarks and open questions

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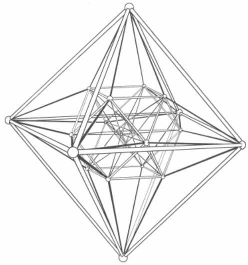
- The only regular polytopes in $\mathbb{R}^n$ other than the n-simplex, n-cube, and n-orthoplex are the dodecahedron and icosahedron in $\mathbb{R}^3$ and three special polytopes in $\mathbb{R}^4$: the 24-cell, 120-cell, and 600-cell. The 24-cell is self-dual while the 120-cell and the 600-cell are dual to each other.



The 24-cell is bounded by 24 octahedra, has 24 vertices, 96 triangular faces and 96 edges.

The 24-cell is the unique self-dual regular Euclidean polytope which is neither a polygon nor a simplex

- In each $\mathbb{R}^n$ there is a generalisation of the cube tiling. The only regular tilings other than cube tilings are two regular tilings of $\mathbb{R}^2$ – the dual tilings by triangles and hexagons – and two dual tilings of $\mathbb{R}^4$, by 24-cells and 4-orthoplexes.

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- What kind of isolated singularities can occur for higher-dimensional periodic minimal hypersurfaces? Are they necessarily algebraic?

Radial eigencubics

Recall that $\Delta_1 u = |\nabla u|^2 \Delta u - \sum_{i,j=1}^{n} u''_{ij} u'_i u'_j$, $u : \mathbb{R}^n \to \mathbb{R}$.

Definition

A polynomial solution of $\Delta_1 u \equiv 0 \mod u$ is called an **eigenfunction**.

An eigenfunction of $\deg u = 3$ is called an **eigencubic**.

A solution of $\Delta_1 u(x) = \lambda |x|^2 \cdot u$ is called a **radial** eigencubic.

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V.T. (2010)

- Any radial eigencubic if harmonic;
- The cubic trace identity holds: $\operatorname{tr}(D^2 u)^3 = -3(n_1 - 1)\lambda u(x)$, where $n_1 \in \mathbb{Z}_{\geq 0}$;
- Any radial eigencubic belongs to either of the following families:
  - a Clifford type eigencubic (an infinite family associated with Clifford symmetric systems);
  - exceptional eigencubics (for example, the Cartan isoparametric cubics in $\mathbb{R}^n$, $n = 5, 8, 14, 26$); only finitely many members exist:

n_1	1	2	3	5	9	0	1	2	4	0	1	2	5	9	0	1	3	1	3	7
n_2	0	0	0	0	0	5	5	5	5	8	8	8	8	8	14	14	14	26	26	26
n	2	5	8	14	26	9	12	15	21	15	18	21	30	42	27	30	36	54	60	72
								?				?	?	?			?	?	?	?

A short introduction into Jordan algebras

A short introduction into Jordan algebras

P. Jordan (1932): a program to discover a new algebraic setting for quantum mechanics by capture intrinsic algebraic properties of Hermitian matrices.

A Jordan (J.) algebra

J is vector space with a bilinear **commutative** product $\bullet : J \times J \to J$ satisfying the **the Jordan identity**

$$x^2 \bullet (x \bullet y) = x \bullet (x^2 \bullet y)$$

The algebra J is **formally real** if additionally $x^2 + y^2 = 0$ implies $x = y = 0$.

For any $x \in J$, the subalgebra $J(x)$ generated by x is associative. The **rank** of J is $\max\{\dim J(x) : x \in J\}$ and the **minimum polynomial** of x is

$$m_x(\lambda) = \lambda^r - \sigma_1(x)\lambda^{r-1} + \ldots + (-1)^r \sigma_r(x) \qquad \text{such that } m_x(x) = 0.$$

$\sigma_1(x) = \mathrm{Tr}\, x$ = the **generic trace** of x,
$\sigma_n(x) = N(x)$ = the **generic norm** (or generic determinant) of x.

Example 1. An associative algebra becomes a J. algebra with $x \bullet y = \frac{1}{2}(xy + yx)$.

Example 2. The Jordan algebra of $n \times n$ matrices over $\mathbb{R}$: $\mathrm{rank}\, x = n$, $\mathrm{Tr}\, x = \mathrm{tr}\, x$, $N(x) = \det x$.

Formally real Jordan algebras

Classification of formally real Jordan algebras

P. Jordan, J. von Neumann, E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, *Annals of Math.*, **1934**

Any (finite-dimensional) formally real J. algebra is a direct sum of the simple ones:

- the algebra $\mathfrak{h}_n(\mathbb{F}_1)$ of symmetric matrices over the reals;
- the algebra $\mathfrak{h}_n(\mathbb{F}_2)$ of Hermitian matrices over the complexes;
- the algebra $\mathfrak{h}_n(\mathbb{F}_4)$ of Hermitian matrices over the quaternions;
- the spin factors $\mathfrak{J}(\mathbb{R}^{n+1})$ with $(x_0, x) \bullet (y_0, y) = (x_0 y_0 + \langle x, y \rangle;\ x_0 y + y_0 x)$;
- $\mathfrak{h}_3(\mathbb{F}_8)$, the Albert exceptional algebra.

In particular, the only possible formally real J. algebras J with $\operatorname{rank}(J) = 3$ are:

a Jordan algebra	**the norm of a trace free element**
$J = \mathfrak{h}_3(\mathbb{F}_d),\ d = 1, 2, 4, 8$	$\sqrt{2}N(x) = u_d(x)$
$J = \mathbb{R} \oplus \mathfrak{J}(\mathbb{R}^{n+1})$	$\sqrt{2}N(x) = 4x_n^3 - 3x_n\lvert x\rvert^2$
$J = \mathbb{F}_1^3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	$\sqrt{2}N(x) = x_2^3 - 3x_2 x_1^2$

Remark. V.T. (2012): the second column is exactly the only cubic solutions to

$$\lvert Du(x)\rvert^2 = 9\lvert x\rvert^4.$$

and provided an explicit correspondence between cubic solutions of the latter equation and formally real cubic Jordan algebras.

Hessian algebras

Definition

A Hessian (non-associative in general) algebra is a vector space V over $\mathbb{F}$ with a non-degenerate scalar product $\langle \cdot, \cdot \rangle$, a cubic form $N : V \to \mathbb{F}$ ($\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$), and a multiplication on V defined by

$$\langle x \# y, z \rangle = N(x; y; z) = \partial_x \partial_y \partial_z N.$$

Notice that the multiplication is necessarily is associative:

$$\langle xy, z \rangle = \langle x, yz \rangle.$$

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Theorem (V.T., 2012)

A cubic form u is a radial eigencubic on $V = \mathbb{R}^n$ if and only if the Hessian algebra on V associated with u possesses the following relation:

$$x^2 \cdot x^2 + 4x \cdot x^3 - 4|x|^2 x^2 - 16N(x)x = 0, \qquad u(x) := \frac{1}{6}\langle x^2, x \rangle,$$

for any element $x \in V$.

Cubic cones via Jordan algebras

Theorem (V.T., 2012])

Let V be the Hessian algebra associated with a minimal radial cubic u.

Then it contains a naturally embedded non-trivial Jordan algebra J.

The radial eigencubic u is Clifford if and only if the associated Jordan algebra J is reduced.

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Theorem (V.T., 2012])

Let V be a cubic Jordan algebra and W be a subspace of V. Assume that there exists a basis $\{e_i\}_{1\le i\le n}$ of W such that

(a) $\sum_{i=1}^n e_i^\# \in W^\perp$;

(b) the linear mapping $\alpha(v) \doteq \sum_{i=1}^n T(v;e_i)e_i : V \to W^\perp$ commute with the adjoint map: $\alpha(v^\#) = (\alpha(v))^\# \mod W^\perp$.

Then the generic norm $N(x) = N(\sum_i x_i e_i)$ is a radial eigencubic in $\mathbb{R}^n$ satisfying

$$\Delta_1 N(x) = -2T(\alpha(x);x)\cdot N(x). \qquad (4)$$

Thank you!