The resultant on compact Riemann surfaces and the exponential transform

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In honour of Emilio Bujalance

The exponential transform and quadrature domains

- $\Omega :=$ a *quadrature domain* (for analytic functions) if

$$\int_\Omega h\,dxdy = \sum_{i=1}^{n} c_i h(z_i) \quad \forall h \in L^1(\Omega), \quad \text{for fixed } z_i \in \Omega, c_i \in \mathbb{C}.$$

- By Richardson's theorem, the complex moments preserved under Hele-Shaw flow:

$$\frac{d}{dt}\int_{\Omega_t} z^k\,dz \wedge d\bar{z} = 0, \quad \forall k \geq 1.$$

- The exponential transform of a bounded closed set Ω is defined by

$$E_\Omega(z,w) = \exp\Big(-\frac{1}{\pi}\int_\Omega \frac{dA(\zeta)}{(\zeta - z)(\bar{\zeta} - \bar{w})}\Big) = 1 - \sum_{m,n=0}^{\infty} \frac{b_{m,n}}{z^{m+1}\bar{w}^{n+1}}.$$

where the moments $a_{m,n} = \iint_\Omega \zeta^m \bar{\zeta}^n\,dA(z)$ a recovered by

$$\sum_{m,n=0}^{\infty} \frac{b_{m,n}}{z^{m+1}\bar{w}^{n+1}} = 1 - \exp\Big(-\sum_{m,n=0}^{\infty} a_{m,n} z^{m+1}\bar{w}^{n+1}\Big).$$

- $E_\Omega(z,w) = 1 - \frac{1}{\bar{w}} C_\Omega(z) + \mathcal{O}(\frac{1}{|w|^2})$, as $|w| \to \infty$, where $C_\Omega(z) = \frac{1}{2\pi i}\int_\Omega \frac{d\zeta \wedge d\bar{\zeta}}{z - \zeta}$ is the Cauchy transform of Ω.

The exponential transform and quadrature domains

The following conditions are equivalent (Aharonov, Shapiro, 1976; Gustafsson, 1983, Putinar, 1996):

- $C_\Omega(z)$ is rational ($:= R(z)$) outside Ω;
- $E_\Omega(z,w)$ is rational $= \frac{Q(z,w)}{P(z)\overline{P(w)}}$, $|z|, |w| \gg 1$;
- Ω is a quadrature domain;
- $\exists S(z)$ meromorphic in Ω: $S(z) = \bar{z}$ on $\partial\Omega$,

$$S(z) = \bar{z} - C_\Omega(z) + R(z), \qquad z \in \Omega.$$

- Ω is determined by finitely many moments a_{jk};
- $\det(b_{jk})_0^N = 0$ for some N.
- There is a bounded linear operator T acting on a Hilbert space, with spectrum equal to Ω, with rank one self commutator $[T^*, T] = \xi \oplus \xi$ and such that the linear span $(T^{*k}\xi)_{k\geq 0}$ is finite dimensional.

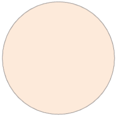
Some properties and examples

- $E(z,z)$ can be viewed as equation of the boundary:

$$E(z,z) = \exp\left[-\frac{1}{\pi}\int\limits_{\Omega}\frac{dA(\zeta)}{|\zeta - z|^2}\right] = \begin{cases} 0, & \text{on} \quad \partial\Omega; \\ > 0, & \text{outside } \Omega, \end{cases}$$

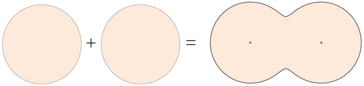
- if Ω_1 and Ω_2 are disjoint then $E_{\Omega_1 \cup \Omega_2} = E_{\Omega_1} E_{\Omega_2}$.

Example 1: the unit disk $\Omega = \mathbb{D}(0,1)$:



$$C_{\mathbb{D}} = \frac{1}{z}, \qquad S(z) = \frac{1}{z}, \qquad E_{\mathbb{D}} = 1 - \frac{1}{z\bar{w}}.$$

Example 2: $\Omega = \mathbb{D}(-1,r) \oplus \mathbb{D}(1,r)$, $r > 1$:



$$C_{\Omega}(z) = \frac{2r^2 z}{z^2 - 1}, \quad E_{\Omega} = 1 - \frac{1 + A(r) z\bar{w}}{(\bar{w}^2 - 1)(z^2 - 1)}.$$

The exponential transform as a resultant

Let Ω be a quadrature domain and let S be the associated Schwarz function of $\partial\Omega$.

Then

$$f(\zeta) = (\zeta, \overline{S(\zeta)}), \quad g(\zeta) = (S(\zeta), \bar{\zeta})$$

are meromorphic on the Schottky double

$$\widehat{\Omega} = \Omega \cup \partial\Omega \cup \widetilde{\Omega}.$$

Theorem (B. Gustafsson, V.T., *Comm. Math. Phys.*, 2009)

The exponential transform $E_\Omega(z, w)$ of a QD can be viewed as the resultant (an *elimination function*) on the Schottky double $\widehat{\Omega}$:

$$E_\Omega(z, w) = \mathcal{R}(f - z, g - \bar{w}) \equiv \mathcal{E}_{f,g}(z, \bar{w}).$$

The polynomial resultant

Given two polynomials

$$P(z) = P_0 + P_1 z + \ldots + P_m z^m = P_m \prod_{i=1}^{m} (z - p_i)$$

$$Q = Q_0 + Q_1 z + \ldots + Q_n z^n = Q_n \prod_{j=1}^{n} (z - q_j)$$

the classical (polynomial) **resultant** is defined by either of the following relations:

$$\begin{aligned}
\mathcal{R}_{\mathrm{pol}}(P, Q) &= P_m^n Q_n^m \prod_{i,j} (p_i - q_j) \\
&= \prod_{i=1}^{m} Q(p_i) = (-1)^{mn} \prod_{j=1}^{n} P(q_j) \\
&= \det \begin{pmatrix}
P_0 & P_1 & \ldots & & P_m & & & \\
& P_0 & P_1 & \ldots & & P_m & & \\
& & \ddots & \ddots & & & \ddots & \\
& & & P_0 & P_1 & \ldots & \ldots & P_m \\
Q_0 & Q_1 & \ldots & & Q_n & & & \\
& Q_0 & Q_1 & \ldots & & Q_n & & \\
& & \ddots & \ddots & & & \ddots & \\
& & & Q_0 & Q_1 & \ldots & & Q_n
\end{pmatrix}
\end{aligned}$$

The polynomial resultant

- **The elimination property**: $\mathcal{R}_{\rm pol}(P,Q)=0 \Leftrightarrow P$ and Q have a common zero.
- **Skew-symmetry**:
$$\mathcal{R}_{\rm pol}(P,Q)=(-1)^{mn}\,\mathcal{R}_{\rm pol}(Q,P).$$
- **Multiplicativity**:
$$\mathcal{R}_{\rm pol}(P_1\cdot P_2,Q)=\mathcal{R}_{\rm pol}(P_1,Q)\cdot\mathcal{R}_{\rm pol}(P_2,Q).$$
- **The Fisher-Hartwig formula:** let $q_0\neq 0$ and $\frac{P(z)}{Q(z)}=\sum_{k=0}^{\infty}s_k z^k$. Then for any $k\geq n=\deg Q$
$$\mathcal{R}_{\rm pol}(P,Q)=p_m^{n-k}q_0^{m+k}\begin{vmatrix} s_m & s_{m-1} & \cdots & s_{m-k+1}\\ s_{m+1} & s_m & \cdots & s_{m-k+2}\\ \vdots & \vdots & \ddots & \vdots\\ s_{m+k-1} & s_{m+k-2} & \cdots & s_m\end{vmatrix}$$

The polynomial resultant

Recent development:

- elimination algebra Jouanolou (1991);
- A-resultants: Gelfand, Kapranov Zelevinsky (1994);
- resultants via Koszul complex by Chardin (1993);
- A "differential resultant" (for commuting differential operators), due to E. Previato (1991);
- toric geometry, resultants and residues, Cattani, Cox, Dickenstein, (1995).

The meromorphic resultant

Let M be a compact Riemann surface and let f, g be meromorphic functions on M. Denote by

$$(f) = f^{-1}(0) - f^{-1}(\infty) = \sum a_i - \sum b_i, \qquad (g) = g^{-1}(0) - g^{-1}(\infty) = \sum c_j - \sum d_j$$

their (principal) divisors.

Definition

Suppose that $x \to \mathrm{ord}_x f \cdot \mathrm{ord}_x g$ is semi-definite on M. Then the **meromorphic resultant** of f is defined by

$$\mathcal{R}(f,g) := g((f)) = \frac{g(f^{-1}(0))}{g(f^{-1}(\infty))}$$

$$= \prod_{x \in M} g(x)^{\mathrm{ord}_x(f)}$$

$$= \prod_{i=1}^{m} \frac{g(a_i)}{g(b_i)}$$

Example: M = the Riemann sphere

Let $M = \mathbb{P}^1$ and

$$f(z) = \lambda \prod_{i=1}^{m} \frac{z-a_i}{z-b_i}, \qquad g(z) = \mu \prod_{j=1}^{n} \frac{z-c_j}{z-d_j}.$$

- **The product formula:**

$$\mathcal{R}(f,g) = \prod_{i=1}^{m}\prod_{j=1}^{n} \frac{a_i-c_j}{a_i-d_j}\cdot\frac{b_i-d_j}{b_i-c_j} = \prod_{i=1}^{m}\prod_{j=1}^{n}(a_i,b_i,c_j,d_j)$$

  where $(a,b,c,d) := \frac{a-c}{a-d}\cdot\frac{b-d}{b-c}$ is the classical cross ratio of two points.

- **Symmetry:** $\mathcal{R}(f,g) = \mathcal{R}(g,f)$ when $M = \mathbb{P}^1$. We shall see that this holds true in general.

- **Homogeneity of degree 0:** $\mathcal{R}(f,g)$ depends only on divisors.

- **Multiplicativity**: $\mathcal{R}(f_1 f_2, g) = \mathcal{R}(f_1,g)\,\mathcal{R}(f_2,g)$.

- **The elimination property**: if f and g are admissible on M then $\mathcal{R}(f,g) = 0$ iff f and g have a common zero or a common pole. In particular, $\mathcal{R}(f,g) = 0$ if f and g are polynomials.

The general meromorphic resultant

In general, by the **Weil reciprocity law** (1940):

$$\prod_{i=1}^{m} \frac{g(a_i)}{g(b_i)} = \prod_{j=1}^{n} \frac{f(c_j)}{f(d_j)}$$

one has the symmetry relation:

$$\mathcal{R}(f, g) = \mathcal{R}(g, f).$$

Integral representation:

$$\mathcal{R}(f, g) = \exp[\frac{1}{2\pi \mathrm{i}} \int_M \frac{df}{f} \wedge d\,\mathrm{Log}\ g],$$

where the latter integral representing $\mathcal{R}(f, g)$ over M the integrand vanishes outside a one-dimensional set of singularities.

Potential theoretic interpretations

Recall that the mutual energy between two signed measures ("charge distributions") μ and ν with compact support in $\mathbb{C}$ is

$$I(\mu,\nu) = -\iint \log|z-\zeta|\, d\mu(z)d\nu(\zeta) = \int U^{\mu} d\nu,$$

where

$$U^{\mu}(z) = -\int \log|z-\zeta|\, d\mu(\zeta).$$

Now, notice that

$$|\mathcal{R}(f,g)|^2 = \exp[\frac{1}{2\pi \mathrm{i}}\int_M \frac{df}{f}\wedge\frac{d\bar{g}}{\bar{g}}].$$

(the expression is a true two-dimensional integral). Then if $d\mu = \delta_{(f)}dx\wedge dy$ and $d\nu = \delta_{(g)}dx\wedge dy$ are regarded as charge distributions then, up to constant factors,

$$I(\mu,\nu) = -\log|\mathcal{R}(f,g)|$$

is the mutual energy between μ and ν.

Elimination function

With $z, w \in \mathbb{C}$ free variables, consider the **elimination function**:

$$\mathcal{E}_{f,g}(z,w) := \mathcal{R}(f-z, g-w) = \frac{(g-w)(f^{-1}(z))}{(g-w)(f^{-1}(\infty))} = \frac{\prod_{i=1}^{m}(g(f_i^{-1}(z))-w)}{\prod_{i=1}^{m}(g(f_i^{-1}(\infty))-w)}$$

Theorem (B. Gustafsson, V.T., 2009)

Let f and g be meromorphic and have no common poles. Then the elimination function is a rational function of the form

$$\mathcal{E}_{f,g}(z,w) = \frac{Q(z,w)}{P(z)R(w)}, \qquad (1)$$

where Q, P, R are polynomials. Notice that $\mathcal{E}_{f,g}$ may be well-defined even if $\mathcal{R}(f,g)$ is not defined.

Corollary: The elimination property

$$\mathcal{E}(f(\zeta), g(\zeta)) = 0 \quad (\zeta \in \mathcal{M}).$$

In particular,

$$Q(f,g) = 0,$$

i.e., the classical polynomial relation between two functions on a compact Riemann surface.

Extended elimination function

Suppose f and g be *arbitrary* meromorphic functions and let us consider the *extended* elimination function of four complex variables:

$$\mathcal{E}_{f,g}(z,w;z_0,w_0) = \mathcal{R}\left(\frac{f-z}{f-z_0}, \frac{g-w}{g-w_0}\right),$$

which is now well-defined and is also rational.

Example

Let f be any meromorphic function of order n and $g = f$. Then

$$\mathcal{E}_{f,f}(z,w;z_0,w_0) = (z,w;z_0,w_0)^n,$$

where $(z,w;z_0,w_0)$ is the cross-product, while $\mathcal{E}_{f,f}(z,w)$ is not defined at all.

Remark. If $\mathcal{E}_{f,g}(z,w)$ is well-defined then

$$\mathcal{E}(z,w;z_0,w_0) = \frac{\mathcal{E}(z,w)\mathcal{E}(z_0,w_0)}{\mathcal{E}(z,w_0)\mathcal{E}(z_0,w)},$$

and in the other direction,

$$\lim_{z_0,w_0\to\infty} \mathcal{E}(z,w;z_0,w_0) = \mathcal{E}(z,w).$$

Main example

Ω is a quadrature domain and, $M = \widehat{\Omega} = \Omega \cup \partial\Omega \cup \widetilde{\Omega}$, and

$$f(\zeta) = (\zeta, \overline{S(\zeta)}), \quad g(\zeta) = (S(\zeta), \bar{\zeta}).$$

Consider

$$f(\zeta) - z, \qquad g(\zeta) - \bar{w}$$

for $\zeta \in \widehat{\Omega}$, $|z|, |w| \gg 1$. Then

$$\mathcal{E}_{f,g}(z, \bar{w}) = \exp[\frac{1}{2\pi \mathrm{i}} \int_M \frac{df(\zeta)}{f(\zeta) - z} \wedge d\,\mathrm{Log}\,(g(\zeta) - \bar{w})]$$

(contributions only from jumps $\Rightarrow$ only from Ω)

$$= \exp[\frac{1}{2\pi \mathrm{i}} \int_\Omega \frac{d\zeta}{\zeta - z} \wedge d\,\mathrm{Log}\,(S(\zeta) - \bar{w})]$$

$$= \exp[-\frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \frac{d\zeta}{\zeta - z} \wedge \mathrm{Log}\,(S(\zeta) - \bar{w})]$$

$$= \exp[-\frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \frac{d\zeta}{\zeta - z} \wedge \mathrm{Log}\,(\bar{\zeta} - \bar{w})]$$

$$= \exp[\frac{1}{2\pi \mathrm{i}} \int_\Omega \frac{d\zeta}{\zeta - z} \wedge \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}}]$$

$$= E_\Omega(z, w),$$

as claimed.

Applications: the determinantal representations

Let $a \in L^\infty(\mathbb{T})$ and $T(a) : H^2 \to H^2$ be the Toeplitz operator acting on the Hardy space H^2:

$$T(a) : \phi \to P_+(a\phi),$$

($P_+ : L^2 \to H^2$ is the orthogonal projection).

Theorem (B. Gustafsson, V.T., 2009)

Let f and g be rational functions such that $|(f)| \subset \mathbb{D}$, $|(g)| \subset \mathbb{C} \setminus \mathbb{D}$. Then

$$\begin{aligned} \mathcal{R}(f,g) &= \det T(f/g)T(g/f) = \det[T(f)^{-1}T(g)T(f)T(g)^{-1}] \\ &= \lim_{N\to\infty} \left(\frac{g(\infty)g(0)}{f(\infty)}\right)^N \cdot \det t_N\left(\frac{f}{g}\right), \end{aligned} \tag{2}$$

where

$$\det t_N(a) \equiv \begin{vmatrix} a_0 & a_{-1} & \dots & a_{1-n} \\ a_1 & a_0 & \dots & a_{2-n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \dots & a_0 \end{vmatrix}, \qquad a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} a(e^{i\theta})d\theta.$$

Remark. The Szegö strong limit theorem.

Applications: A punctured quadrature domain

Let $\Omega = P(\mathbb{D})$, where $P(\zeta) = a_1\zeta + \ldots + a_n\zeta^n$, then for all small enough u, v:

$$E_\Omega(z, w) = \frac{1}{z^n \bar{w}^n} \mathcal{R}_{\text{pol}}(P - z, P^* - \bar{w}\zeta^n)$$

$$= \det \begin{pmatrix} -\frac{1}{z} & & & \bar{a}_n & & \\ a_1 & \ddots & & \vdots & \ddots & \\ \vdots & & -\frac{1}{z} & \bar{a}_1 & & \bar{a}_n \\ a_n & & a_1 & -\frac{1}{\bar{w}} & & \vdots \\ & \ddots & \vdots & & \ddots & \bar{a}_1 \\ & & a_n & & & -\frac{1}{\bar{w}} \end{pmatrix},$$

where

$$P^*(\zeta) = \bar{a}_n + \bar{a}_{n-1}\zeta + \ldots + \bar{a}_1\zeta^{n-1}.$$

Applications: Rational morphisms

Theorem (B. Gustafsson, V.T., 2009)

Let $F(\zeta) : \Omega_1 \to \Omega_2$ be a p-valent proper rational mapping. Then

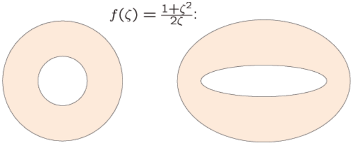
(i) for all $z, w \in \mathbb{C} \setminus \overline{\Omega}_2$

$$E^p_{\Omega_2}(z,w) = \mathcal{R}_u(F(u) - z, \mathcal{R}_v(F(v) - w, E_1(u,v)))$$

where $\mathcal{R}_u$ denote the resultant in u-variable.

(ii) If additionally E_{Ω_1} is a rational function then $E^p_{\Omega_2}$ is also rational.

Example: let $1 < a < b$ and $\Omega = f(\mathbb{D}(a) \setminus \mathbb{D}(b))$ be the confocal elliptic domain,



then

$$E_\Omega = \frac{a^4}{b^4} \cdot \frac{(b^4-1)^2 - 4b^2(1+b^4)z\bar{w} + 4b^4(z^2+\bar{w}^2)}{(a^4-1)^2 - 4a^2(1+a^4)z\bar{w} + 4a^4(z^2+\bar{w}^2)}.$$

Applications: Polynomial vs Meromorphic

Note that the meromorphic resultant for polynomials degenerates: $\mathcal{R}(A,B)=0$. A naïve way to correct this 'defect' is to assign the "value at infinity":

$$B(\infty)=z^n B(1/z)|_{z=0}=B_m$$

and use the original definition

$$\mathcal{R}(A,B):=\prod_{i=1}^{m}\frac{B(a_i)}{B(\infty)}=\prod_{i=1}^{m}\prod_{j=1}^{n}(a_i-b_j),$$

which is consistent with the new definition. In practice, the role of such a 'blow-up' plays the local (or tame) symbol introduced by Serre by

$$\tau_x(f,g):=(-1)^{\mathrm{ord}_x f\,\mathrm{ord}_x g}\cdot\frac{f^{\mathrm{ord}_x g}}{g^{\mathrm{ord}_x f}}(x)\neq 0$$

- $\tau_x(f,g)=1$ a.e. in M;
- Weil's reciprocity law: on a *compact* M,

$$\prod_{x\in M}\tau_x(f,g)=1.$$

Applications: Polynomial vs Meromorphic

A pair (f,g) := *admissible* on $A \subset M$ if the function

$$x \to \mathrm{ord}_x(g)\mathrm{ord}_x(f)$$

is sign semi-definite in A. For examples, polynomials on $\mathbb{P}^1 \setminus \{\infty\}$.
Let f and g be admissible on $M \setminus \{\xi\}$ and ω is a local coordinate near ξ, $\omega(\xi) = 0$. The **reduced** resultant:

$$\mathcal{R}_\omega(f,g) = \tau_\xi(\omega, f)^{\mathrm{ord}_\xi g} \prod_{x \neq \xi} g(x)^{\mathrm{ord}_x(f)},$$

Main example: $M = \mathbb{P}^1$, $A = \mathbb{P}^1 \setminus \{\infty\}$, $\xi = \infty$, $\omega(z) = \frac{1}{z}$. Then for any polynomial $A(z) = A_0 + A_1 z + \ldots + A_m z^m$:

$$\tau_\xi(\omega, A) = \lim_{z \to \infty} \frac{z^m}{A(z)} = \frac{1}{A_m},$$

where $m = -\mathrm{ord}_\infty(A) = \deg A$. Hence

$$\mathcal{R}_\omega(A,B) = A_m^n \prod_{x \neq \infty} B(x)^{\mathrm{ord}_x(A)}$$

coincides with the classical definition:

$$\mathcal{R}_{z,\infty}(A,B) = \mathcal{R}_{\mathrm{pol}}(A,B).$$