Explicit examples of minimal hypersurfaces and algebraic minimal cones

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- 2D minimal surface theory relies heavily on the Weierstrass-Enneper representation and complex analysis tools (uniqueness theorem, reflection principle etc).
- The codimension two case is also very distinguished: any complex hypersurface in $\mathbb{C}^n = \mathbb{R}^{2n}$ is always minimal.
- The only known explicit examples of complete minimal hypersurfaces in $\mathbb{R}^n$, $n \geq 3$, are the catenoids, and minimal hypercones (in particular, the isoparametric ones).
- There are also known to exist some minimal graphs in $\mathbb{R}^n$, $n \geq 9$ (E. Bombieri, de Giorgi, E. Giusti, L. Simon), the (immersed) analogues of Enneper's surface by J. Choe in $\mathbb{R}^n$ for $4 \leq n \leq 7$; the embedded analogues of Riemann one-periodic examples due to S. Kaabachi, F. Pacard in $\mathbb{R}^n$, $n \geq 3$, Scherk's examples due to Pacard. None of the latter examples are known explicitly.
- W.Y. Hsiang (1967): find an appropriate classification of minimal hypercones in $\mathbb{R}^n$, at least of cubic minimal cones.
- V.T. (2012): It turns out that the most natural framework for studying cubic minimal cones is *Jordan algebras* (non-associative structures frequently appeared in connection with elliptic type PDE's); will be discussed later.

Part I: The Additive Ansats

Let $\mathscr{M}$ be a minimal hypersurface in $\mathbb{R}^n$ given by

$$\phi_1(x_1) + \phi_2(x_2) + \phi_3(x_3) + \ldots + \phi_n(x_n) = 0. \tag{1}$$

For $n = 3$: J. Weingarten (1887) (see also Sergienko, V.T., 1998): except for some trivial cases (planes, catenoids, helicoids), the only solutions are those obtained from (ϕ_1, ϕ_2, ϕ_3) satisfying

$$\phi_i'^2 = a_i e^{-2\phi_i} + b_i + c_i e^{2\phi_i}$$

and the coefficients are subject to the rank one condition:

$$\mathrm{rk} \begin{pmatrix} \frac{1}{2}(b_2+b_3) & a_3 & a_2 \\ c_3 & \frac{1}{2}(b_1+b_3) & a_1 \\ c_2 & c_1 & \frac{1}{2}(b_1+b_2) \end{pmatrix} \leq 1$$

This yields several (parametric) families of singly-, doubly- and triply-periodic minimal surfaces in $\mathbb{R}^3$

The Additive Ansats, $n \geq 4$

For $n = 4$ some well-known examples of minimal hypersurfaces in $\mathbb{R}^4$ are:

- a hyperplane, $a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 0$ $\quad \phi_i'^2 = 1$
- the Clifford cone I: $\ln x_1 + \ln x_2 - \ln x_3 - \ln x_4 = 0,$ $\quad \phi_i'^2 = e^{\pm 2\phi_i}$
- the Clifford cone II: $x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0;$ $\quad \phi_i'^2 = \pm 4\phi_i$
- the 3D-catenoid: $x_1^2 + x_2^2 + x_3^2 - \frac{1}{\mathrm{sl}^2(x_4)} = 0;$ $\quad \phi_4'^2 = 4\phi_4(\phi_4^2 - 1)$

Theorem 1 (J. Choe, J. Hoppe, V.T., 2016)

The only minimal hypersurfaces in $\mathbb{R}^n$ satisfying (1) for $n \geq 5$ are the corresponding families above (hyperplanes, quadratic cones and higherdimensional catenoids). For $n = 4$ there are two more, new solutions in $\mathbb{R}^4$:

(i) $\phi_i'^2 = \epsilon_i \cosh \phi_i$, $\epsilon = (-1, -1, 1, 1)$, *one Clifford cone type singularity;*

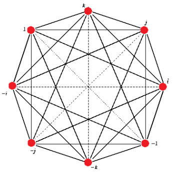
(ii) $\phi_i'^2 = \epsilon_i \sinh \phi_i$, $\epsilon = (-1, -1, 1, 1)$, *infinity many singularities, 4-periodic.*

The four-fold periodic minimal hypersurface in $\mathbb{R}^4$

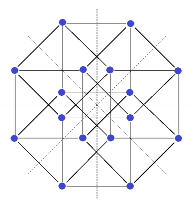
Three remarkable lattices in $\mathbb{R}^4 \cong \mathbb{H}$

Identify $x \in \mathbb{R}^4$ with the quaternion $x_1\mathbf{1} + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4 \in \mathbb{H}$.

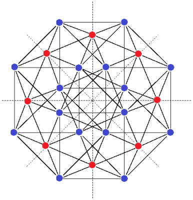
- the *checkerboard lattice*: $D_4 = \{m \in \mathbb{Z}^4 : \sum_{i=1}^4 m_i \equiv 0 \mod 2\}$
- the *Lipschitz integers*: $\mathbb{Z}^4 = \{m \in H : m_i \in \mathbb{Z}\} = D_4 \sqcup (\mathbf{1} + D_4)$
- the *Hurwitz integers* $\mathcal{H} = \mathbb{Z}^4 \sqcup (\mathbf{h} + \mathbb{Z}^4)$, where $\mathbf{h} = \frac{1}{2}(\mathbf{1} + \mathbf{i} + \mathbf{j} + \mathbf{k})$, (the densest possible lattice packing of balls in $\mathbb{R}^4$)



$\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$
the 16-cell



$\frac{1}{2}(\pm\mathbf{1} \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})$
the 8-cell (the hypercube)



...taken together
the 24-cell

(after John C. Baez, *Bull. Amer. Math. Soc.*, 2002)

The construction

Let $S(x) := s(x_1)s(x_2) - s(x_3)s(x_4)$, where $s(t) := \mathrm{sn}(\varpi t, \sqrt{-1})$ is the *lemniscatic sine*, $\varpi = \frac{\Gamma(\frac{1}{4})^2}{2\sqrt{2\pi}}$.

Theorem 2.

$M := \{x \in \mathbb{R}^4 : S(x) = 0\}$ is a connected minimal four-fold periodic embedded hypersurface in $\mathbb{R}^4$ with isolated singular points at the lattice $\mathrm{Sing}(M) = \mathbb{Z}^4 \sqcup (\mathbf{h} + D_4)$ such that

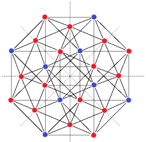
- the 'checkerboard structure': the translating symmetry group of M is D_4
- the orthogonal symmetry group of $M^\bullet$ is $\mathcal{D}_4 \times \mathbb{Z}_2^4$, where $\mathcal{D}_4$ is the dihedral group

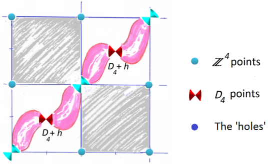
- Singularities of $\mathbb{Z}^4$-type: if $a \in \mathbb{Z}^4$ then

$$S(a+x) = \pm x_1x_2 \pm x_3x_4 + O(|x|^4), \quad \text{as } x \to 0.$$

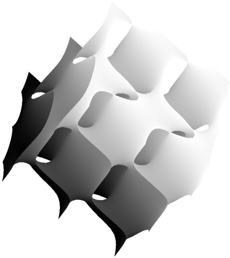
- Singularities of D_4-type: if $a =\in \mathbf{h} + D_4$ then

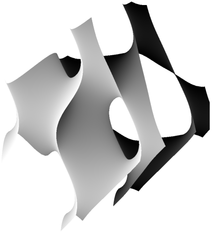
$$S(a+x) = \pm(x_3^2 + x_4^2 - x_1^2 - x_2^2) + O(|x|^4), \quad \text{as } x \to 0,$$

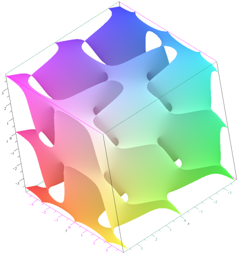


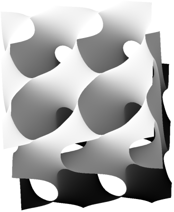


Some cross-section chips of M









Clifford cone singularities

Let

$$\Phi(x) := \frac{s(x_1)s(x_2) - s(x_3)s(x_4)}{(1 + s^2(\frac{x_1+x_2}{2})s^2(\frac{x_3+x_4}{2})) \cdot (1 + s^2(\frac{x_1-x_2}{2})s^2(\frac{x_3-x_4}{2}))}$$

and

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Then $A^2 = I$ (A is a *reflection* in $x_1 - x_2 - x_3 - x_4 = 0$), $B^3 = I$, and

- $\Phi^{-1}(0) = M$
- M is invariant under the A-action: $\Phi(Ax) = \Phi(x) \Rightarrow AM = M$
- $BM = -\mathbf{1} + M$, $B^2 M = -\mathbf{h} + M$
- A is a reflection in $\mathbb{R}^4$ leaving invariant the 'holes' $D_4 + \mathbf{h}^2$.

Question: Does there exist an explicit quaternionic representation of $\Phi(x)$ (as a $\mathbb{H}$-theta series, for example).

Some remarks and speculations

- Characterize embedded minimal submanifolds of $\mathbb{R}^n$ with isolated singularities. L. Caffarelli, R. Hardt, L. Simon (1984): the existence of (bordered) embedded minimal hypersurfaces (non-cones) in $\mathbb{R}^n$, $n \geq 4$, with one isolated singularity. N. Smale (1989): the existence of stable embedded minimal hypersurfaces with boundary, in $\mathbb{R}^n$, $n \geq 8$, with an arbitrary number of isolated singularities.

- One can show that there is no regularly embedded minimal hypersurfaces in $\mathbb{R}^4$ with exactly the same symmetry group as M. Does there exist a D_4-periodic embedded *non-singular* minimal hypersurface in $\mathbb{R}^4$?
- Do there exist minimal hypersurfaces in $\mathbb{R}^8$ and in $\mathbb{R}^{24}$ with E_8 and the Leech lattice symmetry groups resp.?

- Is it possible to glue minimal cones along periodic lattices in $\mathbb{R}^n$ as skeletons to obtain complete embedded (periodic) minimal hypersurfaces?

- What kind of isolated singularities can occur for higher-dimensional periodic minimal hypersurfaces? Are they necessarily algebraic?

Part II: Cubic minimal cones

Cartan isoparametric cubics

- The set $u_1^{-1}(0) \cap S^4 \subset \mathbb{R}^5$ is an (isoparametric) minimal submanifold, where

$$u_1(x) = x_5^3 + \frac{3}{2}x_5(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2) + \frac{3\sqrt{3}}{2}(x_4(x_2^2 - x_1^2) + 2x_1x_2x_3).$$

É. Cartan (1938): u_1 and its counterparts in $\mathbb{R}^8$, $\mathbb{R}^{14}$ and $\mathbb{R}^{26}$

$$u_d := \frac{3\sqrt{3}}{2}\begin{vmatrix} x_2 - \frac{1}{\sqrt{3}}x_1 & \bar{z}_1 & \bar{z}_2 \\ z_1 & -x_2 - \frac{1}{\sqrt{3}}x_1 & \bar{z}_3 \\ z_2 & z_3 & \frac{2}{\sqrt{3}}x_1 \end{vmatrix}, \quad (x, z) \in \mathbb{R}^{3d+2} \cong \mathbb{R}^2 \times \mathbb{A}_d^3, \tag{2}$$

($\mathbb{A}_1 = \mathbb{R}$, $\mathbb{A}_2 = \mathbb{C}$, $\mathbb{A}_4 = \mathbb{H}$, $\mathbb{A}_8 = \mathbb{O}$ the Hurwitz algebras) are the only cubic polynomial solutions of

$$|Du(x)|^2 = 9|x|^4, \qquad \Delta u(x) = 0, \quad x \in \mathbb{R}^n.$$

- $u_d^{-1}(t) \cap S^{3d+1} \subset \mathbb{R}^{3d+2}$ ($t \in [-1, 1]$) is an isoparametric foliation by hypersurfaces with exactly 3 *distinct constant principal curvatures*.
- $u_d(x) = \sqrt{2}N(x)$ on the trace free subspace of the formally real Jordan algebra $J = \mathfrak{h}_3(\mathbb{A}_d)$, $d = 1, 2, 4, 8$. A general fact also holds (V.T., *J. Algebra*, 2015)
- N. Nadirashvili, S. Vlăduț, V.T. (2011): $u_d(x)$ give rise to unusual (singular) viscosity solutions of a uniformly elliptic equation $F(D^2u) = 0$.

… More cubic cones

- $u = \mathrm{Re}(z_1 z_2) z_3$, $z_i \in \mathbb{A}_d$, the triality polynomials in $\mathbb{R}^3, \mathbb{R}^6, \mathbb{R}^{12}$ and $\mathbb{R}^{24}$
- $u = \mathrm{Re}(z_1 z_2) z_3$, $z_i \in \mathrm{Im}\,\mathbb{A}_8$ in $\mathbb{R}^{21}$
- $u = \det x$, where $x \in \mathscr{H}_4'(\mathbb{C}) \cong \mathbb{R}^{15}$, a Hsiang cone (a cubic member thof a Pfaffian family constructed recently by Hoppe-Linardopoulos-Turgut, 2016).
- $u = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{vmatrix}$ in $\mathbb{R}^9$, a member of the determinant family by V.T., 2009.
- $u = (x_1^2 - x_2^2) y_1 + 2 x_1 x_2 y_2 = \langle x, A_1 x \rangle y_1 + \langle x, A_2 x \rangle y_1$, (Lawson's cubic cone),

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In general (V.T. 2010): if $A_i^2 = I$ and $A_i A_j + A_j A_i = 0$, $i \neq j$ then

$$u_A(z) = \sum_{i=1}^{q} \langle x, A_i x \rangle y_i, \quad z = (x, y) \in \mathbb{R}^{2p} \times \mathbb{R}^q$$

is a cubic minimal cone. The existence of a symmetric Clifford system is equivalent to

$$q - 1 \leq \rho(p),$$

where $\rho(p)$ is the Hurwitz-Radon function ($= 1+$ the number of vector fields on $\mathbb{S}^{p-1}$)

- All u satisfy

$$\Delta_1 u(x) := |\nabla u|^2 \Delta u - \sum_{i,j=1}^{n} u_{ij} u_i u_j = \lambda |x|^2 \cdot u \qquad \text{(the Hsiang equation)}$$

- All u are generic norms on a suitable cubic Jordan algebra ...WHY?

The dichotomy of Hisang cubics

Definition. A Hsiang cubic u is said to be of **Clifford type** if $u \cong u_A$ up to an orthogonal transformation; otherwise, it is called **exceptional**.

Representation theory of Clifford algebras yields a complete classification of Hsiang cubics of Clifford type.

How to determine all exceptional Hsiang cubics?

Proposition. *Isoparametric Hsiang cubics, the cubics in* $\mathbb{R}^{15}$ *and* $\mathbb{R}^{21}$ *are exceptional Hsiang cubics.* Nevertheless, the triality cubics in $\mathbb{R}^3, \mathbb{R}^6, \mathbb{R}^{12}$ and $\mathbb{R}^{24}$ are of Clifford type (in fact are *mutants*)

The main results (an analytical point of view)

Main Theorem, Part I

If u is a cubic homogeneous polynomial solution of

$$|Du(x)|^2\Delta u(x) - \tfrac{1}{2}\langle Du(x), D|Du(x)|^2\rangle = \lambda |x|^2 u(x)$$

then

- either $\Delta u(x) = 0$ or u is trivial (depends on one variable, $\sim x_1^3$)
- the cubic trace identity holds:

$$\operatorname{tr}(D^2u)^3 = 3\lambda(n_1 - 1)u, \qquad n_1 \in \mathbb{Z}^+$$

- $n_2 = \frac{1}{2}(n + 1 - 3n_1) \in \mathbb{Z}^+$
- $u(x)$ is exceptional Hsiang cubic iff $n_2 \neq 2$ and the quadratic trace identity holds

$$\operatorname{tr}(D^2u)^2 = C|x|^2, \quad C \in \mathbb{R}$$

The main results (an analytical point of view)

Main Theorem, Part II

There exists finitely many isomorphy classes of exceptional Hsiang algebras.

n	2	5	8	14	26	9	12	15	21	15	18	21	24	30	42	27	30	33	36	51	54	57	60	72
n_1	1	2	3	5	9	0	1	2	4	0	1	2	3	5	9	0	1	2	3	0	1	2	3	7
n_2	0	0	0	0	0	5	5	5	5	8	8	8	8	8	8	14	14	14	14	26	26	26	26	26

In the realizable cases (uncolored):

- If $n_2 = 0$ then $u = \frac{1}{6}\langle z, z^2\rangle$, $z \in \mathscr{H}_3'(\mathbb{A}_d)$, $d = 0, 1, 2, 4, 8$.
- If $n_1 = 0$ then $u(z) = \frac{1}{12}\langle z^2, 3\bar{z} - z\rangle$, $z \in \mathscr{H}_3(\mathbb{A}_d)$, $d = 2, 4, 8$.
- If $n_1 = 1$ then $u(z) = \mathrm{Re}\langle z, z^2\rangle$, $z \in \mathscr{H}_3(\mathbb{A}_d) \otimes \mathbb{C}$, $d = 1, 2, 4, 8$.
- If $(n_1, n_2) = (4, 5)$ then $u = \frac{1}{6}\langle z, z^2\rangle$, $z \in \mathscr{H}_3(\mathbb{O}) \ominus \mathscr{H}_3(\mathbb{R})$

$\mathscr{H}_3(\mathbb{A}_d)$ is the Jordan algebra of 3×3-hermitian matrices over the Hurwitz algebra $\mathbb{A}_d$

Key steps of the proof: a nonassociative algebra approach

u is a solution of a PDE	$\Rightarrow$	a metrized algebra $V(u)$ with an identity

A commutative nonassociative algebra V with an inner product $\langle , \rangle$ is called **metrized** if the multiplication operator $L_x y := xy$ is **self-adjoint**, i.e.

$$\langle xy, z\rangle = \langle x, yz\rangle, \qquad \forall x, y, z \in V.$$

The Freudenthal-Springer construction: given a cubic form u, define an algebra by

$$u(x) = \frac{1}{6}\langle x, x^2\rangle \quad \Leftrightarrow \quad x \cdot y := (D^2u(x))y$$

In this setting,

- the algebra $V = V(u)$ is metrized
- $Du(x) = \frac{1}{2}x^2$
- $L_x = D^2u(x)$, i.e. the multiplication operator by x is the *Hessian* of u at x

Key steps of the proof: a nonassociative algebra approach

Let $u(x)$ be a Hsiang cubic, i.e.

$$|Du(x)|^2 \Delta u(x) - \tfrac{1}{2}\langle Du(x), D|Du(x)|^2\rangle = \lambda |x|^2 u(x)$$

and let $V = V(u)$ be the corresponding Freudenthal-Springer algebra. Then

$$\langle x^2, x^2\rangle \operatorname{tr} L_x - \langle x^2, x^3\rangle = \tfrac{2}{3}\lambda \langle x, x\rangle \langle x^2, x\rangle$$

Def. A metrized commutative algebra is called Hsiang if the latter identity satisfied.

The correspondence: if V is a Hsiang algebra then $u(x) = \frac{1}{6}\langle x, x^2\rangle$ is a Hsiang cubic. In the converse direction, if $u(x)$ is a Hsiang cubic then $V(u)$ is a Hsiang algebra.

Theorem A (The Dichotomy)

- Any nontrivial Hsiang algebra is harmonic: $\operatorname{tr} L_x = 0$.
- u is a Hsiang cubic of Clifford type iff $V(u)$ admits a non-trivial $\mathbb{Z}_2$-grading $V = V_0 \oplus V_1$ such that $V_0 V_0 = 0$.

Key steps of the proof: a nonassociative algebra approach

The set of idempotents of $V(u)$ is nonempty: any maximum point of $u(x)$ on $\mathbb{S}^{n-1}$ gives rise to an idempotent:

$$Du(x_0) = kx_0 \quad \Leftrightarrow \quad \tfrac{1}{2}x_0^2 = kx_0 \quad \Leftrightarrow \quad c^2 = c \text{ for } c = x_0/2k$$

Given an idempotent $c \in V$, L_c is a self-adjoint. Consider the **Peirce decomposition**

$$V = \bigoplus_{\alpha=1}^{k} V_c(t_\alpha), \qquad V_c(t_\alpha) := \ker(L_c - t_\alpha)$$

A key point is by using the original PDE, to determine the multiplicative properties of the Peirce decomposition:

$$V_c(t_\alpha)V_c(t_\beta) \subset \bigoplus_{\gamma} V_c(t_\gamma)$$

If the PDE is 'good enough', there are some hidden (e.g., Clifford or Jordan) algebra structures inside V.

Key steps of the proof: a nonassociative algebra approach

Theorem B (The hidden Clifford algebra structure)

Let V be a Hisang algebra. Then

(i) given an idempotent $c \in V$, the associated Peirce decomposition is

$$V = V_c(1) \oplus V_c(-1) \oplus V_c(-\tfrac{1}{2}) \oplus V_c(\tfrac{1}{2}), \quad \dim V_c(1) = 1;$$

(ii) the Peirce dimensions $n_1 = \dim V_c(-1)$, $n_2 = \dim V_c(-\frac{1}{2})$ and $n_3 = \dim V_c(\frac{1}{2})$ do not depend on a particular choice of c and

$$n_3 = 2n_1 + n_2 - 2;$$

(iii) the following obstruction holds:

$$n_1 - 1 \leq \rho(n_1 + n_2 - 1),$$

where ρ is the Hurwitz-Radon function.

Key steps of the proof: a nonassociative algebra approach

The Peirce decomposition

Setting $V_0 = V_c(1), \quad V_1 = V_c(-1), \quad V_2 = V_c(-\frac{1}{2}), \quad V_3 = V_c(\frac{1}{2})$ we have

	V_0	V_1	V_2	V_3
V_0	V_0	V_1	V_2	V_3
V_1	V_1	V_0	V_3	$V_2 \oplus V_3$
V_2	V_2	V_3	$V_0 \oplus V_2$	$V_1 \oplus V_2$
V_3	V_3	$V_2 \oplus V_3$	$V_1 \oplus V_2$	$V_0 \oplus V_1 \oplus V_2$

In particular, $V_0 \oplus V_1$ and $V_0 \oplus V_2$ are subalgebras of V.

Jordan algebras

An algebra V with a **commutative** product $\bullet$ is called Jordan if

$$[L_x, L_{x^2}] = 0 \qquad \forall x \in V.$$

Main examples

1) The Jordan algebra $\mathscr{H}_n(\mathbb{A}_d)$ of Hermitian matrices of order n, $d = 1, 2, 4$ with

$$x \bullet y = \tfrac{1}{2}(xy + yx)$$

2) The spin factor $\mathscr{S}(\mathbb{R}^{n+1})$ with $(x_0, x) \bullet (y_0, y) = (x_0 y_0 + \langle x, y\rangle;\ x_0 y + y_0 x)$

Theorem (Jordan-von Neumann-Wigner, 1934)

Any finite-dimensional *formally real* Jordan algebra is a direct sum of the simple ones:

- the spin factors $\mathscr{S}(\mathbb{R}^{n+1})$;
- the Jordan algebras $\mathscr{H}_n(\mathbb{A}_d)$, $n \geq 3$, $d = 1, 2, 4$;
- the Albert algebra $\mathscr{H}_3(\mathbb{A}_8)$.

Key steps of the proof

Theorem C (The hidden Jordan algebra structure)

Let V be a Hisang algebra. For any idempotent $c \in V$, the subspace

$$J_c := V_c(1) \oplus V_c(-\tfrac{1}{2})$$

carries a structure of a formally real rank 3 Jordan algebra, and the following conditions are equivalent:

(i) the Hsiang algebra V is *exceptional*;

(ii) J_c is a *simple* Jordan algebra;

(iii) $n_2 \neq 2$ and the *quadratic trace identity* $\operatorname{tr} L_x^2 = c|x|^2$ holds for some $c \in \mathbb{R}$.

The proof of the first part of the theorem is heavily based on the McCrimmon-Springer construction of a cubic Jordan algebra.

The proof of the finiteness of exceptional Hsiang algebras

- Let V be an exceptional Hsiang algebra. Then $J_c := V_c(1) \oplus V_c(-\frac{1}{2})$ is simple formally real Jordan algebra of rank≤ 3 and $\dim J_c = 1 + n_2$.

- The Jordan-von Neumann-Wigner classification implies that either $\dim J_c = 1$ or $\dim J_c = 3d + 3$, where $d \in \{1, 2, 4, 8\}$. Thus, $n_2 = 0$ or $n_2 = 3d + 2$.

- Using the obstruction

$$n_1 - 1 \leq \rho(n_1 + n_2 - 1)$$

and the fact that $\rho(m) \sim \ln m$ implies the finiteness and the possible values in the table.

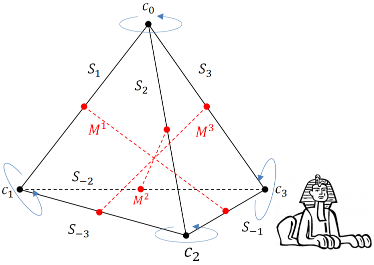
Towards a finer classification

Theorem D. (The tetrad decomposition)

Let V be an exceptional Hsiang algebra, $n_2 = 3d + 2$. Then

$$V = S^1 \oplus S^2 \oplus S^3 \oplus M^1 \oplus M^2 \oplus M^3, \quad S^\alpha = S_\alpha \oplus S_{-\alpha},$$

- M^α are nilpotent;
- each S_α is a real division algebra isomorphic to $\mathbb{A}_d$;
- Any 'vertex-adjacent' triple $(S_\alpha, S_\beta, S_\gamma)$ is a triality



Thank you!