

# Minimal cubic cones via Clifford algebras

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**Abstract.** In this paper, we construct two infinite families of algebraic minimal cones in  $\mathbb{R}^n$ . The first family consists of minimal cubics given explicitly in terms of the Clifford systems. We show that the classes of congruent minimal cubics are in one to one correspondence with those of geometrically equivalent Clifford systems. As a byproduct, we prove that for any  $n \geq 4$ ,  $n \neq 16k + 1$ , there is at least one minimal cone in  $\mathbb{R}^n$  given by an irreducible homogeneous cubic polynomial. The second family consists of minimal cones in  $\mathbb{R}^{m^2}$ ,  $m \geq 2$ , defined by an irreducible homogeneous polynomial of degree  $m$ . These examples provide particular answers to the questions on algebraic minimal cones in  $\mathbb{R}^n$  posed by Wu-Yi Hsiang in the 1960's.

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## 1. Introduction

In 1916, S. Bernstein proved his famous theorem asserting that any entire solution  $u = u(x_1, x_2)$  of the minimal surface equation

$$\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0, \quad (1)$$

must be an affine function. In general, the left hand side of (1) is the mean curvature of the graph  $x_n = u(x_1, \dots, x_{n-1})$  in  $\mathbb{R}^n$ . The question whether the Bernstein result holds true in any dimension  $n \geq 3$  (Bernstein's problem) was a long standing problem until J. Simons [18] proved that the Bernstein property holds in lower dimensions  $n \leq 8$  and E. Bombieri, E. Di Giorgio and E. Giusti in [2] constructed a non-affine minimal graph over  $\mathbb{R}^9$ . An important role in establishing of Bernstein's problem and the constructing of non-affine examples for  $n \geq 9$  played Simon's

cone  $\{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x|^2 = |y|^2\}$  in  $\mathbb{R}^8$  and its generalizations in  $\mathbb{R}^{p+q}$  given by the implicit equation

$$(q-1)(x_1^2 + \dots + x_p^2) + (p-1)(y_1^2 + \dots + y_q^2) = 0, \quad p, q \geq 2. \quad (2)$$

We refer the interested reader to [17], [15] for the further details, and mention a very recent discussion of the breakdown of Bernstein's theorem and geometry of quadratic minimal cones in context of critical dimensions for the stable cone solution of brane in the braneblack-hole system [7], [6].

The Clifford-Simons cones being defined by a quadratic equation are the simplest examples of *algebraic* minimal cones. It is well known that any minimal quadratic cone can be brought into the form (2) in some orthogonal coordinates in  $\mathbb{R}^n$ . In particular, any quadratic minimal cone is completely determined by a non-ordered integer pair  $(p, q)$ .

On the other hand, finding classification of minimal algebraic cones of higher degrees remains a long-standing problem [16], [9], [5]. Even in the case of minimal *cubic* cones there are only few examples known. For our further convenience, we give a short description of these examples.

*Example 1.* By using the representation theory of the compact Lie groups  $SO(3)$  and  $SU(3)$ , one gets two homogeneous minimal cubic cones in  $\mathbb{R}^4$  and  $\mathbb{R}^7$  respectively. For example, the defining equation of the first cone is  $x_3(x_1^2 - x_2^2) + 2x_4x_1x_2 = 0$ . This cone is also a member of the Lawson family [11] of compact minimal surfaces in the unit sphere  $\mathbb{S}^3 \subset \mathbb{R}^4$ .

*Example 2.* Another series of minimal cubic cones is obtained from four isoparametric surfaces with three constant principal curvatures discovered by É. Cartan [3] in 1939. These correspond to standard Veronese embeddings of a projective plane  $\mathbb{F}_dP^2$  into the unit sphere in  $\mathbb{R}^{3d+2}$ ,  $d = 1, 2, 4, 8$ , where  $\mathbb{F}_d$  is one of the four only possible classical division algebras:  $\mathbb{F}_1 = \mathbb{R}$ ,  $\mathbb{F}_2 = \mathbb{C}$ ,  $\mathbb{F}_4 = \mathbb{H}$  and  $\mathbb{F}_8 = \mathbb{O}$ . The corresponding defining polynomials are given explicitly by (cf. [3, p. 34])

$$\begin{aligned} f_d(x) = & x_n^3 - 3x_nx_{n-1}^2 + \frac{3}{2}x_n(X_0\bar{X}_0 + X_1\bar{X}_1 - 2X_2\bar{X}_2) \\ & + \frac{3\sqrt{3}}{2}x_{n-1}(X_0\bar{X}_0 - X_1\bar{X}_1) + \frac{3\sqrt{3}}{2}((X_0X_1)X_2 + \bar{X}_2(\bar{X}_1\bar{X}_0)), \end{aligned}$$

where  $x = (X_0, X_1, X_2, x_{n-1}, x_n)$ , the vectors  $X_k = (x_{kd+1}, \dots, x_{kd+d})$  are identified with the corresponding elements of  $\mathbb{F}_d$ ,  $k = 0, 1, 2$ , and  $\bar{X}$  denotes the conjugate of  $X$  in  $\mathbb{F}_d$ . It is well known that  $f_d^{-1}(0)$  are minimal cubic cones in  $\mathbb{R}^n$ ,  $n = 3d+2$  (thus,  $n = 5, 8, 14$  and  $26$ ).

*Example 3.* W. Y. Hsiang [9] gave an elegant construction of two minimal cubics, in  $\mathbb{R}^9$  and  $\mathbb{R}^{15}$  respectively. The first example exploits some special properties of the orthogonal invariants of the space  $\mathfrak{G}'(4, \mathbb{R})$  of quadratic forms of 4 real variables with trace zero. By identifying  $\mathfrak{G}'(4, \mathbb{R})$  with  $\mathbb{R}^9$ , the defining polynomial of the first cubic is given by  $b_3(Y) = 0$ , where

$$\det(Y - tI_{\mathbb{R}^4}) = t^4 + b_2(Y)t^2 + b_3(Y)t + \det Y, \quad Y \in M,$$

and  $I_V$  stands for the identity operator on  $V$ . We discuss an explicit representation of this cubic in Section 5 below. The second cubic is similarly constructed by making use of the space  $\mathfrak{G}'(4, \mathbb{C})$  of Hermitian forms of 4 complex variables with trace zero.

In this paper, we construct two new families of minimal cones. We show that any Clifford system  $A_0, A_1, \dots, A_q$  in  $\mathbb{R}^{2^m}$  generates an irreducible minimal cubic cone in  $\mathbb{R}^{2^{m+q+1}}$ ; see explicit formulas in Section 3 below. This yields a partial answer to the Problem 1 posed earlier by Hsiang in [9]. Namely, we show that for any  $n \geq 4$ ,  $n \neq 16k + 1$ , there is at least one irreducible minimal cubic in  $\mathbb{R}^n$ . In Section 4 we show that the congruence classes of the constructed minimal cubics are in one-to-one correspondence with those of geometrically equivalent Clifford systems.

In Section 5 we describe another family of minimal cones given by irreducible homogeneous polynomials of arbitrary high degree. More precisely, for any integer  $n \geq 2$  we construct a minimal cone in  $\mathbb{R}^{n^2}$  given by an irreducible homogeneous polynomial of degree  $n$ . If  $n = 3$ , the corresponding cone agrees under an isometry with the Hsiang minimal cubic cone in  $\mathbb{R}^9$  mentioned in Example 3 above.

## 2. Preliminaries

Let  $f \in \mathbb{R}[x_1, \dots, x_n]$  be a homogeneous irreducible polynomial with real coefficients of degree  $s \geq 1$ . If we assume that  $\mathcal{F}(f)$  contains regular points of  $f$ , then  $\mathcal{F}(f) := f^{-1}(0)$  defines an  $(n - 1)$ -dimensional cone in  $\mathbb{R}^n$ . Then it is well known (see, for instance, [9]) that the cone  $\mathcal{F}(f) := f^{-1}(0)$  is minimal if and only if

$$L(f) \equiv 0 \pmod{f}, \quad (3)$$

where the normalized mean curvature operator  $L$  is defined by

$$L(f) := |\nabla f|^3 \operatorname{div} \frac{\nabla f}{|\nabla f|} = |\nabla f|^2 \Delta f - \sum_{i,j=1}^n f_{x_i} f_{x_j} f_{x_i x_j}. \quad (4)$$

For linear forms (i.e.  $s = 1$ ) one has  $L(f) \equiv 0$ , which reflects the well-known fact that hyperplanes in  $\mathbb{R}^n$  have zero mean curvature.

Note that the above congruence also is well-defined for general homogeneous polynomials  $f$ , thus we have the following definition.

**Definition 2.1.** A homogeneous polynomial  $f$  satisfying (3) is called an *eigenfunction* of  $L$ . If  $\deg f = 3$  it is called a *minimal cubic*. The ratio  $\lambda(f) := L(f)/f$  will be called the *weight* of  $f$ .

For a minimal cubic, its weight  $\lambda(f)$  is a quadratic form, hence its diagonal form has an invariant meaning (recall that the operator  $L$  is invariant under orthogonal substitutions [9, Lemma 2]). In this paper, we are primarily interested in the case when

$$\lambda(f) = c|x|^2, \quad c \in \mathbb{R}. \quad (5)$$

Any minimal cubic  $f$  satisfying (5) will be called a *radial* minimal cubic. Notice that for a radial minimal cubics, the weight function  $\lambda(f)$  is rotationally invariant and it contains a complete information about all variables of  $f$ , which makes it possible to define the dimension of  $f$  as the dimension of the vector  $x$ .

*Remark 2.2.* The notion of dimension for general eigenfunctions of  $L$  is not well defined because any eigenfunction in  $\mathbb{R}^n$  also is (by adding ‘redundant variables’) an eigenfunction in a larger  $\mathbb{R}^{n+k}$ .

**Definition 2.3.** Two homogeneous polynomials  $f_1$  and  $f_2$  are called *congruent* if  $f_1(x) = cf_2(Ux)$ , where  $U$  is an orthogonal endomorphism of  $\mathbb{R}^n$  and  $c \in \mathbb{R}$ ,  $c \neq 0$ .

**Proposition 2.4.** *Two irreducible cubics  $f_1$  and  $f_2$  in  $\mathbb{R}^n$  are congruent if and only if the corresponding cones  $\mathcal{F}(f_1)$  and  $\mathcal{F}(f_2)$  are congruent.*

The proof is based on the following lemma.

**Lemma 2.5.** *Let  $f$  be an irreducible cubic,  $f \neq 0$ . Then  $\mathcal{F}(f)$  contains a regular point of  $f$ .*

*Proof.* First we notice that in some orthogonal coordinates,  $f$  can be written as follows:

$$f(x) = ax_n^3 + x_n\phi(x_1, \dots, x_{n-1}) + \psi(x_1, \dots, x_{n-1}), \quad a \neq 0. \quad (6)$$

Indeed, the restriction of  $f(x)$  on the unit sphere  $|x|^2 = 1$  is continuous, hence it attains its maximum at some point  $x^0$  (since  $f(x)$  is a non-identically zero odd function, the maximum is strictly positive). By Lagrange principle, the gradient of  $f$  is collinear to the unit vector  $\nabla|x|^2$  at  $x^0$ , i.e.  $\nabla f(x^0) = ax^0$ . We have by the homogeneity of  $f$ ,

$$a = \langle \nabla f(x^0), x^0 \rangle = 3f(x^0) > 0.$$

Setting  $e_n = x^0$  and completing  $e_n$  to an orthonormal basis of  $\mathbb{R}^n$ , one can easily see that  $f(x)$  takes the required gap form (6) in the new coordinates.

We shall proceed by contradiction. Suppose that  $\mathcal{F}(f)$  contains no regular points of  $f$ , i.e.  $f = 0$  implies  $\nabla f = 0$ . Since  $f$  is irreducible,  $\psi \neq 0$ , hence there exists  $u = (u_1, \dots, u_{n-1}) \neq 0$  such that  $\psi(u) \neq 0$ . Then the cubic polynomial  $P(t) \equiv at^3 + bt + c$  has a real root  $t_1 \neq 0$ , where  $b = \phi(u)$  and  $c = \psi(u)$ . Denote by  $T$  the set of all real roots of  $P(t)$ . Then for any  $t_i \in T$ , the point  $(u_1, \dots, u_{n-1}, t_i) \in \mathcal{F}(f)$ , thus  $\nabla f(u_1, \dots, u_{n-1}, t_i) = 0$ . The latter implies that the partial derivative  $\partial_{x_n} f(u_1, \dots, u_{n-1}, t_i) = 0$ . Thus  $P'(t_i) = 0$  for any  $t_i \in T$ , that is any real zero of the polynomial  $P(t)$  must be also a zero of its derivative  $P'(t)$ . But it is possible only if  $P(t) \equiv at^3$ , which contradicts to  $c = \psi(u) \neq 0$ . The lemma is proved.  $\square$

*Proof of Proposition 2.4.* It suffices to prove the ‘only if’ part. Suppose that the cones  $\mathcal{F}(f_1)$  and  $\mathcal{F}(f_2)$  are congruent, i.e.  $\mathcal{F}(f_2) = U\mathcal{F}(f_1)$  for some orthogonal endomorphism  $U$  of  $\mathbb{R}^n$ . Consider the new cubic  $g(x) = f_2(Ux)$ . Then  $g(x) = 0$  whenever  $f_1(x) = 0$ . By Lemma 2.5, the cone  $\mathcal{F}(f_1)$  contains a regular point.

Applying the real Nullstellensatz for the algebraic set  $\mathcal{F}(f_1)$  given by a single polynomial equation (see, for instance, Lemma 2.5 in [12]), we conclude that  $g(x)$  must be a multiple of  $f_1$ . Since  $\deg g = \deg f_1 = 3$ , we conclude that  $g = cf_1(x)$  for some real  $c \neq 0$ . This shows that  $cf_1(x) = f_2(Ux)$ , the theorem follows.  $\square$

### 3. Clifford minimal cubics

We shall exploit the well-known fact that any Clifford algebra has a linear representation in a Euclidean space  $\mathbb{R}^{2m}$  such that all algebra generators act as orthogonal endomorphisms. This link between the representation theory of Clifford algebras and the so-called Clifford systems can be described as follows [1]. Recall that a  $q$ -tuple  $\mathcal{A} = (A_0, \dots, A_q)$ ,  $q \geq 1$ , of symmetric endomorphisms of  $\mathbb{R}^{2m}$  is called a (*symmetric*) *Clifford system* on  $\mathbb{R}^{2m}$ , or  $\mathcal{A} \in \text{Cliff}(\mathbb{R}^{2m}, q)$ , if

$$A_i A_j + A_j A_i = 2\delta_{ij} \cdot I_{\mathbb{R}^{2m}}, \quad 0 \leq i, j \leq q. \quad (7)$$

Recall that by  $I_V$  we denote the identity operator on  $V$ .

*Remark 3.1.* Notice that all  $A_i$  are orthogonal matrices and they are necessarily trace free. Indeed, since the trace is invariant under cyclic permutations, we have from (7) for  $i \neq j$ :  $\text{tr } A_i = -\text{tr } A_j A_i A_j = -\text{tr } A_i A_j^2 = -\text{tr } A_i$ .

Given a finite collection of matrices  $A_i \in \mathbb{R}^{s \times s}$ ,  $0 \leq i \leq q$ , and a vector  $z = (z_0, z_1, \dots, z_q) \in \mathbb{R}^{q+1}$ , we make use the following notation:

$$A_z := \sum_{i=0}^q z_i A_i.$$

Now we are ready to expose the first family of examples announced in the introduction.

**Theorem 3.2 (Clifford cubics).** *Let  $\mathcal{A} = (A_0, \dots, A_q) \in \text{Cliff}(\mathbb{R}^{2m}, q)$ . Then*

$$\Phi(x) \equiv \Phi_{\mathcal{A}}(x) := y^t A_z y, \quad x = (y, z) \in \mathbb{R}^{2m} \oplus \mathbb{R}^{q+1}, \quad (8)$$

*is a radial minimal cubic satisfying (3) with the weight*

$$\lambda(\Phi) = -8|x|^2. \quad (9)$$

*Proof.* By Remark 3.1, all  $A_i$  are trace free. Hence  $\Delta\Phi = 0$  and

$$\Phi_{y_i} = 2e_i^t A_z y, \quad \Phi_{z_i} = y^t A_i y, \quad (10)$$

and  $\Phi_{y_i y_j} = 2e_i^t A_z e_j$  to (4), where  $\{e_i\}_{1 \leq i \leq 2m}$  is the standard orthonormal basis in  $\mathbb{R}^{2m}$  (we interpret  $e_i$  as a vector-column). We have

$$\begin{aligned} -L(\Phi) &= 8 \sum_{i,j=1}^{2m} e_i^t A_z y \cdot e_i^t A_z e_j \cdot e_j^t A_z y + 8 \sum_{i=1}^{2m} \sum_{k=0}^q e_i^t A_z y \cdot e_i^t A_k y \cdot y^t A_k y \\ &=: S_1 + S_2. \end{aligned} \quad (11)$$

We find by virtue of (7) that

$$\begin{aligned} A_{z'} A_{z''} + A_{z''} A_{z'} &= \sum_{i,j=0}^q z'_i z''_j (A_i A_j + A_j A_i) = I_{\mathbb{R}^{2m}} \sum_{i,j=0}^q 2\delta_{ij} z'_i z''_j \\ &= \langle z', z'' \rangle I_{\mathbb{R}^{2m}}, \end{aligned} \quad (12)$$

where  $\langle \cdot, \cdot \rangle$  is the standard scalar product in  $\mathbb{R}^q$ . In particular,  $A_z^2 = |z|^2 I_{\mathbb{R}^{2m}}$ , hence

$$A_z^3 = |z|^2 A_z. \quad (13)$$

On the other hand,  $\sum_{i=1}^{2m} e_i e_i^t = I_{\mathbb{R}^{2m}}$ , thus we obtain

$$S_1 = 8 \sum_{i,j=1}^{2m} y^t A_z (e_i e_i^t) A_z (e_j e_j^t) A_z y = 8 y^t A_z^3 y = 8|z|^2 \cdot y^t A_z y.$$

Now applying (12) to  $z' = z$  and  $z'' = (y^t A_0 y, \dots, y^t A_q y)$ , we find

$$y^t A_z A_{z''} y = \frac{1}{2} y^t (A_z A_{z''} + A_{z''} A_z) y = |y|^2 \langle z, \tau \rangle = |y|^2 \cdot y^t A_z y,$$

which yields

$$S_2 = 8 \sum_{j=0}^q y^t A_z A_j y \cdot y^t A_j y = 8 y^t A_z A_\tau y = 8|y|^2 \cdot y^t A_z y.$$

Substituting the found relations into (11) yields  $L(\Phi) = -8(|z|^2 + |y|^2)\Phi$ , hence (9) is proved.

In order to show that  $\Phi$  is irreducible, we assume the contrary. Then any specialization of  $\Phi$  must be reducible too. For instance, by setting  $z_i = 0$ ,  $2 \leq i \leq q$ , we see that the cubic form

$$z_0 \cdot y^t A_0 y + z_1 \cdot y^t A_1 y \quad (14)$$

is reducible. Since  $A_0$  and  $A_1$  also form a Clifford system, one can choose new orthogonal coordinates  $(u, v) \in \mathbb{R}^m \times \mathbb{R}^m$  such that  $A_0(u, v) = (u, -v)$  and  $A_1(u, v) = (v, u)$ ; see [1, Lemma 5.4.7]. Then setting  $u_i = v_i = 0$ ,  $2 \leq i \leq m$ , in (14) we conclude that the resulted specialization  $g := z_0(u_1^2 - v_1^2) + 2z_1 u_1 v_1$  must be reducible too. The latter expression, however, cannot be reducible even over a bigger complex polynomial ring  $\mathbb{C}[z_0, z_1, u_1, v_1]$  because the discriminant of  $g$  with respect to  $u_1$  is  $4v_1^2(z_0^2 + z_1^2)$ , not a perfect square. The contradiction finishes the proof.  $\square$

**Definition 3.3.** We shall call  $\Phi_{\mathcal{A}}(x)$  a *Clifford minimal cubic*.

It follows from the general theory of Clifford systems (see, for example, [14]) that a necessary and sufficient condition for existence of a symmetric Clifford system  $\mathcal{A}$  in  $\mathbb{R}^{2m}$  of cardinality  $q$  is that  $q \leq \rho(m)$ , where  $\rho$  is the Hurwitz-Radon function

$$\rho(2^s \cdot \text{odd}) = 8a + 2^b, \quad \text{where } s = 4a + b, \quad 0 \leq b \leq 3. \quad (15)$$

Thus, an irreducible Clifford minimal cubic  $\Phi_{\mathcal{A}}$  does exist in  $\mathbb{R}^n$  precisely if equation  $n = 2m + q + 1$  has a solution  $(q, m)$  satisfying  $1 \leq q \leq \rho(m)$ . In that case we say that  $n$  is *realizable*, and the pair  $(q, m)$  is *admissible* for the number  $n$ .

The above results yield a partial answer to Problem 1 posed by Hsiang in [9] on the existence of irreducible cubics in a given dimension  $n \geq 4$ .

**Corollary 3.4.** *Let  $n \geq 4$  and  $n \not\equiv 1 \pmod{16}$ . Then there is at least one irreducible radial minimal cubic in  $\mathbb{R}^n$ .*

*Proof.* It easily follows from the definition of the Hurwitz-Radon function that  $\rho(2^s k) \geq 2^s$  for  $s = 0, 1, 2, 3$  and any integer  $k \geq 1$ .

If  $n$  is even then  $(q, m) = (1, \frac{n-2}{2})$  is admissible for  $n$  because  $\rho(m) \geq 1$  for any  $m \geq 1$ . Thus any even  $n \geq 4$  is realizable.

Now assume that  $n$  is odd. A simple verification shows that there is exactly two non-realizable values of  $n$  for  $n \leq 16$ , namely  $n = 5$  and  $n = 9$ . On the other hand, in  $\mathbb{R}^5$  there is an isoparametric minimal cubic considered in Example 2 and in  $\mathbb{R}^9$  the Hsiang minimal cubic given in Example 3. These cubics are easily shown to be radial.

Thus, we can suppose that  $n$  is odd and  $n \geq 16$ . Write  $n = 16k + s$ , where  $k \geq 1$ ,  $s$  is odd and  $s \leq 15$ . If  $s \in \{3, 5, 7, 9\}$  then the pair  $(s-1, \frac{n-s}{2}) \equiv (s-1, 8k)$  is admissible for  $n$  because one has  $\rho(8k) \geq 8$ .

If  $s \in \{11, 15\}$  then  $n$  has the form  $n = 4p+3$ , hence the pair  $(2, \frac{n-3}{2}) \equiv (2, 2p)$  is admissible for  $n$  because  $\rho(2p) \geq 2$ . If  $s = 13$  then  $n$  has the form  $n = 8p + 5$ , hence the pair  $(4, \frac{n-5}{2}) \equiv (4, 4p)$  is admissible for  $n$  because  $\rho(4p) \geq 4$ . The corollary is proved.  $\square$

*Remark 3.5.* A more delicate argument shows that some terms of the exceptional sequence  $n = 16k + 1$  are really non-realizable (for example, those corresponding to  $k \leq 2^7 = 128$ ). On the other hand, all terms of the form  $n = 2^{11}p + 17$ ,  $p \geq 1$ , are realizable.

**Conjecture 1.** There is no irreducible radial minimal cubics in  $\mathbb{R}^{17}$ .

*Example 4.* For any integer  $m \geq 1$ , the following matrices define a Clifford system on  $\mathbb{R}^{2m}$ :

$$A_0 = \begin{pmatrix} I_{\mathbb{R}^m} & 0 \\ 0 & -I_{\mathbb{R}^m} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & I_{\mathbb{R}^m} \\ I_{\mathbb{R}^m} & 0 \end{pmatrix}.$$

Setting  $y = (x_1, \dots, x_{2m})$ ,  $z = (x_{2m+1}, x_{2m+2})$ , we obtain by Theorem 3.2 the following irreducible radial minimal cubic in  $\mathbb{R}^{2m+2}$ :

$$\Phi_{\mathcal{A}} = x_{2m+1}(x_1^2 + \dots + x_m^2 - x_{m+1}^2 - \dots - x_{2m}^2) + 2x_{2m+2}(x_1x_{m+1} + \dots + x_mx_{2m}).$$

For  $m = 1$ , we get the Lawson minimal cubic mentioned in Example 1 above.

#### 4. Congruent Clifford minimal cubics

Recall that two cones are called congruent if they agree under an orthogonal endomorphism in  $\mathbb{R}^n$ . After the new examples of minimal cubics were shown to exist, a natural question is to characterize all congruent Clifford minimal cubics in  $\mathbb{R}^n$ . The aim of this section is to show that the congruence classes of Clifford minimal cubics are in one to one correspondence with those of the geometrically equivalent Clifford systems.

We begin with mentioning several well-known basic facts about representation theory of Clifford systems [1, Section 5.5]. Recall that two Clifford systems  $\mathcal{A} = (A_0, \dots, A_q)$  and  $\mathcal{B} = (B_0, \dots, B_q)$  in  $\text{Cliff}(\mathbb{R}^{2m}, q)$  are called *geometrically equivalent* if there is an orthogonal endomorphism  $a$  of  $\mathbb{R}^{2m}$  such that

$$S(\mathcal{A}) = a^t S(\mathcal{B}) a, \quad (16)$$

where  $S(\mathcal{A}) = \{A_z : |z| = 1\}$  is the unit sphere in the span( $A_0, \dots, A_q$ ).

A Clifford system  $\mathcal{A}$  on  $\mathbb{R}^{2m}$  is called irreducible if it is not possible to write  $\mathbb{R}^{2m}$  as a direct sum of two non-trivial subspaces that are invariant under all of the  $A_i$ . Then it is well known that

- each Clifford system is geometrically equivalent to a direct sum of irreducible Clifford systems;
- an irreducible Clifford system  $(A_0, \dots, A_q)$  on  $\mathbb{R}^{2m}$  exists precisely when  $m = 2^s$  and

$$\rho\left(\frac{m}{2}\right) < q \leq \rho(m), \quad (17)$$

where  $\rho$  is the Hurwitz-Radon function (15). In particular, for small values  $q$ , the possible pairs  $(q, m)$  are

$q$	1	2	3	4	5	6	7	8	9	10	11
$m$	1	2	4	4	8	8	8	8	16	32	64

- For any  $q \geq 1$  there is exactly one class of geometrically equivalent irreducible systems.

Our first observation is the following formula for the cardinality  $q$  of the Clifford system  $\mathcal{A}$  in  $\Phi_{\mathcal{A}}$ .

**Proposition 4.1.**

$$\tau(\Phi_{\mathcal{A}}) := \frac{|x|^2 \text{tr } H^3(\Phi_{\mathcal{A}})}{24L(\Phi_{\mathcal{A}})} = q - 1, \quad (18)$$

where  $H(f)$  denotes the Hessian matrix of  $f$ .

*Proof.* Write the Hessian matrix of  $\Phi := \Phi_{\mathcal{A}}$  in the block form

$$H(\Phi) \equiv \begin{pmatrix} \Phi_{yy} & \Phi_{yz} \\ \Phi_{zy} & \Phi_{zz} \end{pmatrix} = \begin{pmatrix} 2A_z & Q \\ Q^t & 0 \end{pmatrix},$$



where  $Q$  is the matrix with entries  $Q_{ij} = \Phi_{y_i z_j} = 2e_i^t A_j y$ . Thus

$$\operatorname{tr} H^3(\Phi) = 8 \operatorname{tr} A_z^3 + 6 \operatorname{tr} A_z Q Q^t. \quad (19)$$

By virtue of (13) and Remark 3.1, the first term in (19) is zero. In order to determine the second term, we note that

$$\begin{aligned} \operatorname{tr} A_z Q Q^t &= \sum_{i,j=1}^{2m} (A_z)_{ij} (Q Q^t)_{ij} = \sum_{i,j=1}^{2m} (A_z)_{ij} \sum_{k=0}^q Q_{ik} Q_{jk} \\ &= 4 \sum_{i,j=1}^{2m} (A_z)_{ij} \sum_{k=1}^q e_i^t A_k y \cdot e_j^t A_k y \\ &= 4 \sum_{k=0}^q y^t A_k \left( \sum_{i,j=1}^{2m} (A_z)_{ij} e_i \cdot e_j^t \right) A_k y \\ &= 4 \sum_{k=0}^q y^t A_k A_z A_k y. \end{aligned} \quad (20)$$

Note also that  $A_k^3 = A_k$ , hence on applying (12) we obtain

$$\begin{aligned} \sum_{k=0}^q A_k A_z A_k &= \sum_{k=0}^q (A_k A_z + A_z A_k) A_k - \sum_{k=0}^q A_z A_k^2 \\ &= 2 \sum_{k=0}^q z_k A_k^3 - (q+1) A_z = (1-q) A_z, \end{aligned}$$

which by virtue of (20) and (19) yields

$$\operatorname{tr} H^3(\Phi) = 24(1-q)\Phi. \quad (21)$$

The latter relation yields the required formula (18) by virtue of (9).  $\square$

**Corollary 4.2.** *Consider two Clifford minimal cubics in  $\mathbb{R}^n$*

$$\Phi_{\mathcal{A}}(x) = \sum_{i=0}^q z_i y^t A_i y, \quad \Phi_{\mathcal{B}}(X) = \sum_{j=0}^Q Z_j Y^t B_j Y, \quad (22)$$

where  $x = (y, z) \in \mathbb{R}^{2m} \oplus \mathbb{R}^q \approx \mathbb{R}^n$ ,  $X = (Y, Z) \in \mathbb{R}^{2M} \oplus \mathbb{R}^Q \approx \mathbb{R}^n$ , and  $\mathcal{A} \in \operatorname{Cliff}(\mathbb{R}^{2m}, q)$ ,  $\mathcal{B} \in \operatorname{Cliff}(\mathbb{R}^{2M}, Q)$ .

*If the cubics  $\Phi_{\mathcal{A}}(x)$  and  $\Phi_{\mathcal{B}}(X)$  are congruent then  $q = Q$ .*

*Proof.* Notice that for any homogeneous cubic polynomial  $f$ ,  $\tau(f)$  is an invariant operator under orthogonal transformations and dilatations:

$$\tau(f(x)) = \tau(f(U(x))), \quad \tau(f(cx)) = \tau(f(x)).$$

Indeed, the first property follows from the facts that both the operator  $L$  (see, for example, [9, Lemma 2]) and the trace  $\operatorname{tr} H^3(f)$  are invariant under orthogonal transformations. The second property follows from the homogeneity of  $f$ . Thus  $\tau(\Phi_{\mathcal{A}}) = \tau(\Phi_{\mathcal{B}})$ , which implies by (18) that  $q = Q$ , the corollary is proved.  $\square$

**Theorem 4.3 (Congruence criteria).** *The Clifford cubics  $\Phi_{\mathcal{A}}(x)$  and  $\Phi_{\mathcal{B}}(X)$  given by (22) are congruent in  $\mathbb{R}^n$  if and only if the Clifford systems  $\mathcal{A}$  and  $\mathcal{B}$  are geometrically equivalent.*

*Proof.* We first prove the "if" part. Then  $q = Q$ , and  $(A_0, \dots, A_q)$  and  $(B_0, \dots, B_q)$  are geometrically equivalent. Denote by  $a$  the corresponding orthogonal endomorphism of  $\mathbb{R}^{2m}$  in (16). Then

$$A_i = a^t B_{w_i} a, \quad 0 \leq i \leq q, \quad (23)$$

where  $|w_i| = 1$ ,  $w_i \in \mathbb{R}^{q+1}$ . We have  $B_{w_i} = \sum_{k=0}^q w_{ik} B_k$ , where  $\sum_{k=1}^q w_{ik}^2 = 1$ . By virtue of (23) and (7),

$$\begin{aligned} 2\delta_{ij} I_{\mathbb{R}^{2m}} &= A_i A_j + A_j A_i = \sum_{k,l=0}^q w_{ik} w_{jl} \cdot a^t B_k B_l a \\ &= 2I_{\mathbb{R}^{2m}} \sum_{k,l=0}^q w_{ik} w_{jl} \delta_{kl} = 2I_{\mathbb{R}^{2m}} \sum_{k=0}^q w_{ik} w_{jk}, \end{aligned}$$

thus  $d := (w_{ij})_{0 \leq i, j \leq q}$  is an orthogonal matrix. Furthermore, by (23)

$$A_z \equiv \sum_{k=0}^q z_k A_k = a^t \left( \sum_{k=0}^q w_{ik} z_k B_k \right) a \equiv a^t B_{dz} a, \quad (24)$$

and thus  $y^\top A_z y = (ay)^\top B_{dz} ay$ . Write  $\mathbb{R}^{2m+q+1} \cong V_y \oplus V_z$  according to the vector decomposition  $x = y \oplus z$ . Then  $\Phi_{\mathcal{A}}(x) = \Phi_{\mathcal{B}}(Ux)$ , where  $U$  is an orthogonal endomorphism of  $V_y \oplus V_z$ :

$$U = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},$$

which yields that  $\Phi_{\mathcal{A}}$  and  $\Phi_{\mathcal{B}}$  are congruent.

Conversely, suppose  $\Phi_{\mathcal{A}}$  and  $\Phi_{\mathcal{B}}$  are congruent Clifford minimal cubics given by (22). Then  $q = Q$  by Corollary 4.2, so that the congruence relation (after scaling by a constant factor, if needed) reads as

$$\Phi_{\mathcal{A}}(x) = \Phi_{\mathcal{B}}(Ux), \quad (25)$$

where  $U$  is an orthogonal endomorphism of  $\mathbb{R}^{2m+q+1}$ . Let  $\mathbb{R}^{2m+q+1} = V_y \oplus V_z$  be the decomposition associated with  $\mathcal{A}$  and  $x = y + z$ . Then  $U$  can be written in the block form as follows

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad UU^t = U^t U = I_{\mathbb{R}^n}.$$

It follows from (25) that

$$|\nabla \Phi_{\mathcal{A}}|^2|_{(y,z)} = |\nabla \Phi_{\mathcal{B}}|^2|_{U(y,z)}. \quad (26)$$

By using (10),

$$|\nabla \Phi_{\mathcal{A}}|^2 = 4|A_z y|^2 + \sum_{i=0}^q (y^t A_i y)^2 = 4|y|^2 |z|^2 + \sum_{i=0}^q (y^t A_i y)^2,$$

and on applying (26), the relation (26) reads as follows:

$$4|y|^2|z|^2 + \sum_{i=0}^q (y^t A_i y)^2 = 4|ay + bz|^2 |cy + dz|^2 + \sum_{i=0}^q ((ay + bz)^t B_i (ay + bz))^2. \quad (27)$$

Setting  $z = 0$  in the latter identity yields  $4|bz|^2 |dz|^2 + \sum_{i=0}^q ((bz)^t B_i (bz))^2 = 0$ , which is equivalent to the system

$$4|bz|^2 |dz|^2 = 0, \quad (28)$$

$$(bz)^t B_i (bz) = 0, \quad 0 \leq i \leq q. \quad (29)$$

Since the polynomial ring over  $\mathbb{R}$  contains no zero divisors, (28) implies that either (i)  $bz \equiv 0$  or (ii)  $dz \equiv 0$ .

Consider first (ii). Then  $d \equiv 0$ , and (29) additionally yields for any  $i$ ,  $0 \leq i \leq q$ , that  $v^t B_i v \equiv 0$  for any  $v \in W := b(V_z)$ . Since  $B_i$  is a symmetric endomorphism, we conclude that the restriction  $B_i|_W \equiv 0$ ; in particular,  $B_i bz \equiv 0$  for any  $z \in V_z$ . Since  $d = 0$ , we find from the orthogonality relation  $U^t U = 1_{\mathbb{R}^n}$  that  $a^t b \equiv 0$ . Thus (27) becomes

$$4|y|^2|z|^2 + \sum_{i=0}^q (y^t A_i y)^2 = 4(|ay|^2 + |bz|^2) |cy|^2 + \sum_{i=0}^q ((ay)^t B_i (ay))^2. \quad (30)$$

By homogeneity,  $|y|^2|z|^2 = |bz|^2|cy|^2$ , thus  $b^t b = I_{V_z}$  and  $c^t c = I_{V_y}$ . On the other hand, by the orthogonality relation  $U^t U = 1_{\mathbb{R}^n}$  we have  $a^t a + c^t c = I_{V_y}$ , so that  $a^t a = 0$ . It follows that  $a \equiv 0$  and substituting this into (30) yields

$$\sum_{i=0}^q (y^t B_i y)^2 = 0,$$

a contradiction.

Consider now the remaining alternative (i). Then  $b \equiv 0$  and from the orthogonality relations  $UU^t = U^t U = 1_{\mathbb{R}^n}$  we infer  $aa^t = I_{V_y}$ ,  $c = 0$  and  $d^t d = I_{V_z}$ . Therefore  $a$  (resp.  $d$ ) is an orthogonal endomorphism of  $V_y$  (resp. of  $V_z$ ). Substituting this into (25) yields

$$y^t A_z y = (ay)^t B_{dz} (ay). \quad (31)$$

Since both  $A_z$  and  $B_{dz}$  are symmetric matrices, the latter identity implies  $A_z = a^t B_{dz} a$ . Since  $a$  and  $d$  are orthogonal transformations, we arrive to (16), thus  $\mathcal{A}$  and  $\mathcal{B}$  are geometrically equivalent. The theorem is proved completely.  $\square$

Applying the facts about irreducible Clifford systems given in the beginning of the section, one can determine the number  $c(n)$  of non-congruent minimal Clifford cubics in  $\mathbb{R}^n$ . These numbers for values  $n \leq 21$  are given in the table below.

$n$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$c(n)$	1	0	2	2	2	0	4	4	8	4	6	6	6	0	9	9	18	9

## 5. Minimal cones of arbitrary high degree

In [9, Problem 2], Hsiang asks the following question. For a given dimension  $n \geq 4$ , are there irreducible homogeneous polynomials in  $n$  real variables of *arbitrary high degree*, which give minimal cones of codimension one in  $\mathbb{R}^n$ ? Below we give a partial answer on this question by showing that there are irreducible minimal cones of arbitrary high degree.

Let  $X = (x_{ij})_{1 \leq i, j \leq m}$  be an  $m$ -by- $m$  square matrix,  $m \geq 1$ . Consider the determinantal function

$$\Psi_m(X) = \det(x_{ij})$$

which is an element of the polynomial ring  $\mathbb{R}[x_{11}, \dots, x_{mm}]$ . It is well known that  $\Psi_m(X)$  is a irreducible polynomial in the polynomial ring  $\mathbb{C}[x_{11}, \dots, x_{mm}]$  (see, for example, [22, § 30], [10, p. 630]).

**Theorem 5.1.**  $\Psi_m(X)$  is an irreducible eigenfunction of  $L$  of degree  $m$ .

*Proof.* The case  $m = 1$  is trivial and for  $m = 2$  the theorem is equivalent to the well-known fact that the Clifford cone  $x_{11}x_{22} - x_{12}x_{21} = 0$  is a minimal submanifold in  $\mathbb{R}^4$ . Thus, we may suppose that  $m \geq 3$ .

We have  $\partial_{x_{ij}} \Psi_m = (-1)^{i+j} \det X^{i|j}$ . Here and in what follows,  $X^{\alpha|\beta}$  denotes the submatrix of  $X$  obtained by deleting the rows indexed by  $i \in \alpha$  and columns indexed by  $j \in \beta$  for the index sets  $\alpha, \beta \subset \{1, \dots, m\}$ . Similarly,  $X_{\alpha|\beta}$  denotes the submatrix of  $X$  that lies in the corresponding rows and columns. Note that in the given notation,  $x_{ij} = X_{i|j}$ .

Thus, the second derivative is found as:

$$\partial_{x_{ij} x_{kl}}^2 \Psi_m = (-1)^{i+j+k+l} \epsilon_{i,j;k,l} \cdot \det X^{i,k|j,l}, \quad (32)$$

where  $\epsilon_{i,j;k,l}$  is the sign of  $(i-k)(j-l)$  when the product is nonzero and  $\epsilon_{i,j;k,l} = 0$  otherwise. Since  $\Psi_m$  is linear in each variable,  $\Delta \Psi_m = 0$ , thus

$$\begin{aligned} L(\Psi_m) &= - \sum^* \partial_{x_{ij} x_{kl}}^2 \Psi_m \partial_{x_{ij}} \Psi_m \partial_{x_{kl}} \Psi_m \\ &= - \sum^* \epsilon_{i,j;k,l} \det X^{i,k|j,l} \det X^{i|j} \det X^{k|l} \end{aligned} \quad (33)$$

where  $\sum^*$  denotes the sum over all indices satisfying  $(i-k)(j-l) \neq 0$ .

On the other hand,  $\epsilon_{i,j;k,l} = -\epsilon_{i,l;k,j}$  and  $\det X^{i,k|j,l} = \det X^{i,k|l,j}$ , hence

$$\begin{aligned} L(\Psi_m) &= - \sum^* \partial_{x_{il} x_{kj}}^2 \Psi_m \partial_{x_{il}} \Psi_m \partial_{x_{kj}} \Psi_m \\ &= - \sum^* \epsilon_{i,l;k,j} \det X^{i,k|l,j} \det X^{i|l} \det X^{k|j} \\ &= \sum^* \epsilon_{i,j;k,l} \det X^{i,k|j,l} \det X^{i|l} \det X^{k|j} \end{aligned} \quad (34)$$

thus

$$\begin{aligned} L(\Psi_m) &= -\frac{1}{2} \sum^* \epsilon_{i,j;k,l} (\det X^{i|j} \det X^{k|l} - \det X^{i|l} \det X^{k|j}) \det X^{i,k|j,l} \\ &= -\frac{1}{2} \sum^* \epsilon_{i,j;k,l} \begin{vmatrix} \det X^{i|j} & \det X^{i|l} \\ \det X^{k|j} & \det X^{k|l} \end{vmatrix} \cdot \det X^{i,k|j,l}. \end{aligned} \quad (35)$$

By using the representation of the inverse matrix  $X^{-1}$  in terms of its adjoint,

$$(X^{-1})_{ij} = \frac{(-1)^{i+j}}{\det X} \det X^{ij},$$

we find for the  $2 \times 2$  determinant in (35),

$$\begin{aligned} \begin{vmatrix} \det X^{ij} & \det X^{il} \\ \det X^{kj} & \det X^{kl} \end{vmatrix} &= (-1)^{i+j+k+l} (\det X)^2 \cdot \begin{vmatrix} \det(X^{-1})_{ij} & \det(X^{-1})_{kj} \\ \det(X^{-1})_{il} & \det(X^{-1})_{kl} \end{vmatrix} \\ &\equiv (-1)^{i+j+k+l} (\det X)^2 \cdot \epsilon_{i,j;k,l} \det(X^{-1})_{ik|jl} \end{aligned}$$

On the other hand, there is a formula relating the minors of  $X^{-1}$  to those of  $X$  (see [8, § 0.8.4]), which reads in our notation as follows:

$$\det(X^{-1})_{\alpha|\beta} \det X = (-1)^{|\alpha|+|\beta|} \det X^{\beta|\alpha},$$

where  $|\alpha| = \sum_{i \in \alpha} i$ . Combining this with (35) we obtain

$$L(\Psi_m) = -\frac{\det X}{2} \sum^* (\det X^{ik|jl})^2. \quad (36)$$

Thus, we see that  $\Psi_m$  satisfies (3) with  $\lambda(\Psi_m) = -\frac{1}{2} \sum^* (\det X^{ik|jl})^2$ .  $\square$

*Example 5.* Consider the case  $m = 3$  in Theorem 5.1. Then

$$\Psi_3 = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} = x_1 x_5 x_9 + x_2 x_6 x_7 + x_3 x_4 x_8 - x_3 x_5 x_7 + x_2 x_4 x_9 + x_1 x_6 x_8,$$

and the corresponding weight function is found as:

$$\lambda(\Psi_3) = -\frac{1}{2} \sum^* (\det X^{ik|jl})^2 = -\frac{1}{2} \sum_{i=1}^9 x_i^2.$$

The latter relation shows that  $\Psi_3$  is a radial minimal cubic in  $\mathbb{R}^9$ . In fact, it can be shown by a direct computation that the Hsiang cubic  $b_3(X)$  from Example 3 is congruent to  $\Psi_3$ .

## 6. Concluding remarks

There is some formal resemblance between an appearance of Clifford systems in the above constructions (cf. (8)) and in the well-known examples of the isoparametric hypersurfaces with four principal curvatures given by Ferus, Karcher and Mzünzner in [4]. Recall that the latter are the level-sets in the unit sphere of following quartic polynomials:

$$F(y) = |y|^4 - 2 \sum_{i=0}^q (y^t A_i y)^2, \quad y \in \mathbb{R}^{2m}.$$

Since these isoparametric hypersurfaces generate minimal quartics (as focal varieties), it would be interesting to learn whether there is any reasonable connection between the FKM-quartics and the Clifford minimal cubics constructed on the present paper (cf. [19]).

Another curious observation is that all the irreducible minimal cubics mentioned in Examples 1–3, as well as the new examples of Clifford minimal cubics constructed in Section 3, are the *radial* minimal cubics. On the other hand, *reducible* minimal cubics need not to be radial as the following example shows. Consider a reducible cubic  $f = x_6(2x_1^2 + 2x_2^2 - x_3^2 - x_4^2 - x_5^2)$ . Then it is an eigenfunction with the weight

$$\lambda(f) = -28(x_1^2 + x_2^2) - 10(x_3^2 + x_4^2 + x_5^2) - 16x_6^2.$$

In a forthcoming paper [21] we study the general radial cubics in more detail. Some further observations make the following conjecture plausible.

**Conjecture 2.** Any minimal *irreducible* cubic is radial.

Note also that among all minimal cubics considered here, only isoparametric minimal cubics given in Example 2 are properly immersed, i.e. the corresponding defining polynomial  $f$  satisfies the non-degenerating property  $|\nabla f| \neq 0$  on  $\mathcal{F}(f) \setminus \{0\}$ . It follows from a recent result of O. Perdomo [13] that there are no properly immersed minimal cubics in  $\mathbb{R}^4$ . It would be interesting to know if this property extends to higher dimensions.

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