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On a theorem of S.Y. Cheng and S.T. Yau

Tkachev Vladimir G.

1. Introduction

In their paper [2], Cheng and Yau have established the following result generalizing the classical Bernstein theorem for minimal surface equation.

THEOREM A (Corollary 1 in [2]). *Let $H(t)$ be a function of constant sign on $\mathbb{R}^1$ such that*

$$H'(t) \geq 0. \tag{1}$$

Then any entire solution $f = f(x_1, x_2)$ of the mean curvature equation

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\frac{f_{x_i}}{\sqrt{1 + |\nabla f(x)|^2}} \right) = H(f(x)) \tag{2}$$

is a linear function.

The proof given in [2] exploits a new technique based on Liouville type theorems on Riemannian manifolds with polynomial volume growth assumptions [1]

In this note, we show that Theorem A is a corollary of the classical Bernstein theorem [1] and an elementary capacity estimate given in Theorem 1 below. In fact, our method implies a generalization of Theorem A for weak solutions of a wider class PDEs without any sign assumptions on $H(t)$.

Let us fix some notation. Let D be a domain in $\mathbb{R}^2$ and let $A_i(x, \xi)$, $i = 1, 2$ be a Baire functions defined for any $x = (x_1, x_2) \in D$ and $\xi = (\xi_1, \xi_2)$, and such that

a) $\sum_{i=1}^{2} \xi_i A_i(x, \xi) \geq 0$,

[1]The sign of the derivative $H'(t)$ is important as one can see from the following example: $f(x) = r^2/\sqrt{1 + r^2}$, where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is a (covex) C^∞-solution in $\mathbb{R}^2$ of (2) with $0 < H \leq 4$ and $H'(t) < 0$. Notice that the origin version [2] use another sign convention on H in (2), such that Theorem A in [2] is valid under the nonincreasing condition $\partial H / \partial x_3 \leq 0$ instead.

b) $\sum_{i=1}^2 A_i^2(x,\xi) \le 1$.

A locally Lipschitz function $f(x)$ in D is said to be a (weak) solution of the equation

$$\sum_{i=1}^{2} \frac{d}{dx_i} A_i(x, \nabla f(x)) = 0 \tag{3}$$

if for any Lipschitz function $\varphi(x)$ with a compact support in D the following equality holds:

$$\iint\limits_D \sum_{i=1}^{2} A_i(x, \nabla f(x)) \varphi'_{x_i} \, dx_1 dx_2 = - \iint\limits_D \varphi(x) H(f(x)) \, dx_1 dx_2. \tag{4}$$

It is easy to see that $f(x)$ is a classical solution of (3) in D provided A_i and f are smooth enough.

Further, given a pair of disjoint closed subsets P, Q of $\mathbb{R}^2$, define the capacity of the condenser (P,Q) as

$$\operatorname{cap}(P,Q) = \inf \iint\limits_D |\nabla \varphi(x)|^2 \, dx_1 dx_2,$$

where the infimum is taken over all locally Lipschitz functions $\varphi(x)$ such that $\varphi(x) \equiv 1$ on P and $\varphi(x) \equiv 0$ on Q.

Theorem 1. *Let $H(t)$ be a function subject to the condition (1) and let $f(x)$ be a weak solution of (3) in $D \subset \mathbb{R}^2$. Then for any compact subset $F \Subset D$*

$$\iint\limits_D H^2(f(x)) \, dx_1 dx_2 \le 4 \operatorname{cap}(F, \mathbb{R}^2 \setminus D). \tag{5}$$

Proof. Let $\psi(x)$ be a Lipschitz function with a compact support in D, $\psi \equiv 1$ on F. Then $\phi := \psi^2(x) H(f(x))$ is also a Lipschitz function with a compact support in D. Thus, we have

$$\iint\limits_D \sum_{i=1}^{2} A_i(x, \nabla f) \frac{\partial}{\partial x_i} (\psi^2 H(f)) \, dx_1 dx_2 = - \iint\limits_D \psi^2 H^2(f) \, dx_1 dx_2.$$

Taking into account (1) and the condition (a) above, we arrive at

$$-2 \iint\limits_D \psi H(f) \sum_{i=1}^{2} A_i(x, \nabla f) \frac{\partial \psi}{\partial x_i} \, dx_1 dx_2 \ge \iint\limits_D \psi^2 H^2(f) \, dx_1 dx_2.$$

Using (b), we get

$$2 \iint\limits_D |\psi H(f)| \cdot |\nabla \psi| \, dx_1 dx_2 \ge \iint\limits_D \psi^2 H^2(f) \, dx_1 dx_2,$$

and the Cauchy inequality yields

$$\iint_D \psi^2 H^2(f)\, dx_1 dx_2 \le 4 \iint_D |\nabla \psi|^2\, dx_1 dx_2.$$

Taking into account that $\psi \equiv 1$ on F we arrive at (5). □

As an application, we obtain the following generalization of Theorem A above without requirement of constant sign on H. Indeed, we have for the mean curvature operator (2):

$$A_i(x,\xi) = \frac{\xi_i}{\sqrt{1+\xi_1^2+\xi_2^2}}, \quad i = 1, 2,$$

which obviously satisfies the conditions (a) and (b) above. Let $f(x)$ be a classical solution of (2), where $H'(t) \ge 0$. Using the well-known fact that $\mathbb{R}^2$ has parabolic conformal type, any compact subset $F \Subset \mathbb{R}^2$ has capacity zero, i.e. there exists a sequence of open sets $F \Subset F_i \Subset F_{i+1}$ and $\cup_{i=1}^{\infty} F_i = \mathbb{R}^2$ such that

$$\lim_{i\to\infty} \operatorname{cap}(F, \mathbb{R}^2 \setminus F_i) = 0,$$

we obtain for any entire solution f of (2) by (5) that

$$\iint_F H^2(f(x))\, dx_1 dx_2 = 0$$

for any compact set F, thus $H(f(x)) \equiv 0$ everywhere in $\mathbb{R}^2$. Using the classical Bernstein theorem [1], we get that f is an affine function.

References

1. S. Bernstein, *Über ein geometrisches Theorem und seine Anwendung auf die partiellen Differentialgleichungen vom elliptischen Typus*, Math. Z. **26** (1927), no. 1, 551–558.
2. S. Y. Cheng and S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), no. 3, 333–354.

Department of Mathematics, Linköping University, 581 83 Linköping, Sweden

E-mail address: `vladimir.tkatjev@liu.se`