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GENERALIZATIONS OF COMPLEX ANALYSIS
AND THEIR APPLICATIONS IN PHYSICS II

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LIFE-TIME OF MINIMAL TUBES AND COEFFICIENTS OF UNIVALENT FUNCTIONS IN A CIRCULAR RING

Summary

Estimate of life-time of two-dimensional minimal tubes in $\mathbb{R}^3$ have been obtained via potential theory method. The connection between this problem and coefficients of univalent functions in an annulus have been established.

1. Introduction

Let $x = (x_1, x_2, \ldots, x_n, x_{n+1})$ be a point in Euclidean space $\mathbb{R}^{n+1}$ with the *time* axis Ox_{n+1} and M be a p-dimensional Riemannian manifold, $2 \leq p \leq n$.

Definition 1. We say that a surface $\mathcal{M} = (M, \mathbf{u})$ given by C^2-immersion $\mathbf{u} : M \to \mathbb{R}^{n+1}$ is a *tube* with the projection interval $\tau(\mathcal{M}) \subset Ox_{n+1}$, if (i) for any $\tau \in \tau(\mathcal{M})$ the sections $\Sigma_\tau = f(\mathcal{M}) \cap \Pi_\tau$ by hyperplanes $\Pi_\tau = \{x \in \mathbb{R}_1^{n+1} : x_{n+1} = \tau\}$ are not empty compact sets; (ii) for $\tau', \tau'' \in \tau(\mathcal{M})$ any part of $\mathcal{M}$ situated between two different $\Pi_{\tau'}$ and $\Pi_{\tau''}$ is a compact set.

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Definition 2. A surface $\mathcal{M}$ is called *minimal* if the mean curvature of $\mathcal{M}$ vanishes everywhere.

It is the well known fact (see [5], p.331) that the minimality condition of $\mathcal{M}$ is equivalent to that all coordination functions of the immersion $\mathbf{u}$ are harmonic. For this reason, the two-dimensional minimal tubes can be considered as direct analog of the closed relative string conception in the modern nuclear physics (cf. [2]). This approach was proposed by V.M.Miklyukov and the autor in [7] for an arbitrary dimension p.

From this point of view many intrinsic geometric invariants of $\mathcal{M}$ have the natural physical meaning. Namely, the length of the projection interval $|\tau(\mathcal{M})|$ can be interpreted as a *life-time* of the tube $\mathcal{M}$.

To introduce the following important characteristic we denote by ν the unit normal to Σ_τ with respect to $\mathcal{M}$ which is co-directed with the time-axis Ox_{n+1}. Then by virtue of the harmonicity of the coordinate functions $u_k(m) = x_k \circ \mathbf{u}(m)$, $1 \leq k \leq n+1$, the flow integrals

$$J_k = \int_{\Sigma_\tau} \langle \nabla u_k, \nu \rangle \, d\Sigma$$

are independent of $\tau \in \tau(\mathcal{M})$. Here $d\Sigma$ is the 1-Hausdorff measure along Σ_τ.

Definition 3. We call $Q(\mathcal{M}) = (J_1, J_2, \dots, J_{n+1}) \in \mathbb{R}^{n+1}$ the *full flow-vector* of $\mathcal{M}$.

We notice the positiveness of J_{n+1} as a consequence of the choice of ν direction. Moreover, $Q(\mathcal{M})$ is an 1-homogeneous functional of $\mathcal{M}$ under the homotheties group action in $\mathbb{R}^{n+1}$. Let us denote by $\alpha(\mathcal{M})$ the angle between $Q(\mathcal{M})$ and the time-axis Ox_{n+1}.

In this paper we are intersted in following question: What sufficient conditions yield the finiteness of the time-life of a two-dimensional minimal tube? As it was shown in the series of papers [6] – [8], in the case $p \geq 3$ this quantity is always finite and the following estimation holds

$$|\tau(\mathcal{M})| \leq \varrho(\mathcal{M})\, c_p,$$

where c_p depends only on p, and $\varrho(\mathcal{M})$ is the smallest diameter of sections Σ_τ. The last relationship is sharp and the equality occurs if and only if $\mathcal{M}$ is a minimal surface of revolution.

A special feature of the two-dimensional case is that there exist tubes with finite as well as infinite values of the life-time. A crucial observation for that is an existence of an additional family of the slanting minimal tubes having circular section Σ_τ as against the many-dimensional case. This class of surfaces were discovered by Riemann [10]. Some recent examples can be found in [4].

In this paper we prove

Theorem 1. *Let $\mathcal{M}$, $\dim\mathcal{M} = 2$, be a minimal two-connected tube with univalent Gaussian mapping. If the angle $\alpha(\mathcal{M})$ is different from zero, then the life-time $|\tau(\mathcal{M})|$ of $\mathcal{M}$ is finite and*

$$\tau(\mathcal{M}) \leq \frac{\pi \|Q\| \cos \alpha(\mathcal{M})}{\ln \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)}.$$

Let us denote by $a_0[f]$ the central coefficient of the Laurent decomposition of an holomorphic function $f(z)$ in an annulus $K_R = \{z : 1/R < |z|, R\}$, i.e.

$$a_0[f] \equiv \int_{C_1} \frac{d(\zeta)\, d\zeta}{\zeta}.$$

where C_1 is the unite circle $\{z \in \mathbb{C} : |z| = 1\}$. The following auxiliary assertion is a key ingredient in the proof of Theorem 1.

Theorem 2. *Let $g(z)$ be a univalent holomorphic function defined in the annulus K_R and omitting zero. Assume that*

$$a_0[g] = \lambda, \qquad a_0[1/g] = -\lambda, \tag{1}$$

for some real positive λ. Then

$$\ln R \leq \ln R_0(\lambda) := \frac{\pi^2}{\ln(\lambda + \sqrt{1+\lambda^2})} \tag{2}$$

Remark 1. We note that estimate (2) has well asymptotic behaviour for $R \to \infty$ as shows Riemannian example mentioned above. But we cann't now present the sharp estimate for $\ln R$. Nevertheless, it seemed us very probably that the following conjecture is true.

Conjecture. The best upper bound of the left side of (2) is achieved for the Weierstrass-type holomorphic function $g_0(z)$ which maps the annulus onto the plain $\mathbb{C}$ with two slits: $(-1/\alpha; 0)$ and $(\alpha; +\infty)$, for the suitable choice of parameter α.

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2. Proof of Theorem 2

Let $\Gamma = \{C_\rho : 1/R < \rho < R\}$ be a family of all concentric circles $C_\rho = \{z : |z| = \rho\}$ in the annulus K_R. It follows easily from the non-vanishing property of $g(z)$ that the loop C_1 in the integrals (1) may be replaced by an arbitrary circle $C_\rho \in \Gamma$. It follows from the mean value theorem and (1) that for every $\rho \in (1/R; R)$ there exist t_1 and t_2 such that

$$\operatorname{Re} g(\rho e^{it_1}) = \lambda \quad \text{and} \quad \operatorname{Re} \frac{1}{g(\rho e^{it_2})} = -\lambda. \tag{3}$$

Let $\gamma_\rho = g(C_\rho)$. Then by virtue of the univalence of $g(z)$, the curve γ_ρ is the simple Jordan one. Let $g(\rho e^{it}) = x(t) + iy(t)$ be the representation of γ_ρ. Then we obtain from (3)

$$x(t_1) = \lambda; \qquad x^2(t_2) + y^2(t_2) + \frac{1}{\lambda}x(t_2) = 0.$$

The last relations have the helpful geometric interpretation:

($\star$) The curve γ_ρ intersects the vertical rightline $L_1 = \{z : \operatorname{Re} z = \lambda\}$ and the circle $L_2 = \{z : |z + 1/2\lambda| = 1/2\lambda\}$.

Now we recall the following definition from the potential theory.

Definition 4. Let E be a family of locally rectifiable curves γ and $\varphi(z) \geq 0$ be a Baire function with the property

$$\int_\gamma \varphi(z)\,|dz| \geq 1,$$

for every $\gamma \in E$. The infimum

$$\operatorname{mod} E = \inf \int \varphi^2(z)\,dx\,dy$$

over all such $\varphi(z)$ is called a *conformal module* of the family E.

Then it is known (see [1]) that $\operatorname{mod} E$ is the conformal invariant. As a consequence we obtain in our situation

(4) $$\operatorname{mod} \Gamma = \operatorname{mod} \Gamma_1,$$

where $\Gamma_1 = \{\gamma_\rho : 1/R < \rho < R\}$.

Let us denote by D the two-dimensional domain

$$D = \left\{z : \operatorname{Re} z < \lambda; \left|z + \frac{1}{2\lambda}\right| > \frac{1}{2\lambda}\right\}.$$

Using ($\star$)-property, we can find for every $\rho \in (1/R; R)$ the continuum $\gamma'_\rho \subset \gamma_\rho$ joining the boundary components of D. Then a family Γ_2 consiting of all continua γ'_ρ is "shorter" than Γ_1 and it follows from Theorem 1.2, [1] that

(5) $$\operatorname{mod} \Gamma_1 \leq \operatorname{mod} \Gamma_2.$$

On the other hand, Γ_2 is the subfamily of $\Gamma(D)$, where the last term means the family of *all* curves joining the boundary components of a domain D. The monotonicity property of infimum and Definition 4 lead to the following inequality

(6) $$\operatorname{mod} \Gamma_2 \leq \operatorname{mod} \Gamma(D).$$

Now, combining the standard fact

(7) $$\operatorname{mod} \Gamma = \frac{\ln R}{\pi}$$

with relations (4), (5) and (6) we arrive at the following inequality

$$\frac{\ln R}{\pi} \leq \operatorname{mod} \Gamma(D).$$

To compute the last module we note that the linear-fractional function

$$f(z) = \frac{1}{\lambda^*} \cdot \frac{z + \lambda^*}{1 - z\lambda^*}$$

maps D onto an annulus $K_1 = \{w : 1 < |w| < 1/\lambda^{*2}\}$, where $\lambda^* = \sqrt{\lambda^2 + 1} - \lambda$. Thus, using the invariance property of conformal module we obtain

$$\frac{\ln R}{\pi} \leq \operatorname{mod}(D) \equiv \frac{2\pi}{\ln(1/\lambda^{*2})} = \frac{\pi}{\ln(\lambda + \sqrt{1 + \lambda^2})}.$$

and Theorem 2 is proved.

3. Gaussian map two-dimensional minimal tubes and the full-flow vector

In this section we express the full flow-vector of an arbitrary two-dimensional tube $\mathcal{M} \in \mathbb{R}^n$ via Chern-Weierstrass representation for minimal surfaces. Namely, if $\mathcal{M}$ is a two-connected surface then we can arrange that $\mathcal{M}$ is conformally equivalent to an annulus K_R for the appropriate $R > 1$. Then there exists the corresponding parametrization of $\mathcal{M}$ (see [9]):

$$\mathbf{u}(z) = \operatorname{Re} \int_{z_0}^{z} F(\zeta)\,d\zeta \; : K_R \to \mathbb{R}^n,$$

where

$$F(z) = (\varphi_1(\zeta), \ldots, \varphi_n(\zeta))$$

and $\varphi_i(\zeta)$ are holomorphic functions satisfying the following conditions

(8) $$\sum_{i=1}^{n} \varphi_i(\zeta)^2 = 0$$

and

(9) $$\operatorname{Re} \int_{|z|=1} f(\zeta)\,d\zeta = 0.$$

Lemma 1. *Under the above hypotheses we have*

$$Q(\mathcal{M}) = \mathrm{Im}\int_{|z|=1} F(\zeta)\,d\zeta. \tag{10}$$

Proof. It sufficient to show that

$$J_k \equiv \int_{\Sigma_r} \langle \nabla_{u_k}, \nu\rangle\, d\Sigma = \mathrm{Im}\int_{|z|=1} \varphi_k(\zeta)\,d\zeta, \tag{11}$$

for every $k = 1, 2, ..., n+1$.

To prove (11) we introduce the conjugate to $u_k(z)$ function $v_k(z)$ by

$$v_k^*(z) = \mathrm{Im}\int_{z_0}^{z} \varphi_k(\zeta)\,d\zeta,$$

We notice that $v_k(z)$ in general is a multi-valued function. On the other hand, the covariant derivative ∇_{v_k} is well defined and using the properties of Hodge $\star$ - operator we have

$$\int_{\Sigma_r} \langle \nabla u_k, \nu\rangle\, d\Sigma = \int_{\Sigma_r} \langle \star\nabla u_k, \star\nu\rangle\, d\Sigma = \int_{\Sigma_r} \langle \nabla v_k, \star\nu\rangle\, d\Sigma =$$

$$= \int_{\Sigma_r} dv_k = \mathrm{Im}\int_{|z|=1} \varphi_k(\zeta)\,d\zeta,$$

and (11) is proved.

In our case $n = 2$, Chern-Weierstrass representation can be simplified in the following classic way. Namely, there exist a holomorphic function $f(z)$ and a meromorphic function $g(z)$ which are well defined in the annulus K_R and such that

$$F(z) = ((1-g^2)f;\ i(1+g^2)f;\ 2gf). \tag{12}$$

Moreover, poles of $g(z)$ coincide with zeros of $f(z)$ and the order of a pole of $g(z)$ is precisely the order of the corresponding zero of $f(z)$. We emphasize that $g(z)$ is a composition of the stereographic projection and Guassian map of $\mathcal{M}$.

Lemma 2. *In our assumptions*

$$2fg \equiv \frac{\langle Q(\mathcal{M}), e_3\rangle}{2\pi\, z}, \tag{13}$$

and $g(z)$ omits the zero and infinity values.

Proof. We use the method proposed by M.Schiffman in [11]. We racell that the coordinate function $u_3(z)$ is harmonic in the annulus K_R and by virtue of Definition 1,

$$\lim_{z\to 1/R} u_3(z) = \tau_1, \quad \lim_{z\to R} u_3(z) = \tau_2, \tag{14}$$

where $\tau(\mathcal{M}) = (\tau_1; \tau_2)$ is the projection of the tube $\mathcal{M}$ onto x_3-axis.

We consider an auxiliary harmonic function

$$h(z) = \tau_1 + \frac{\tau_2 - \tau_1}{2\ln R}\ln|z|.$$

It is easily seen that $h(z)$ satisfies (14). Thus $h_1(z) = u_3(z) - h(z)$ is harmonic in the annulus and

$$\lim_{z\to\partial K_R} h_1(z) = 0.$$

Then the maximum principle implies that $h_1(z) \equiv 0$ everywhere in K_R and hance

$$u_3(z) \equiv \tau_1 + \frac{\tau_2 - \tau_1}{2\ln R}\ln|z|. \tag{15}$$

In particular, it follows from (15) that

$$du_3(z) \equiv \frac{\tau_2 - \tau_1}{\ln R}\cdot\frac{z}{|z|^2}$$

doesn't vanish in K_R. We have, as a consequence, the normal $n(z)$ to $\mathcal{M}$ isn't parallel to e_3 at any point. Taking into account the above remark about the geometrical sense of $g(z)$ we obtain that $g(z) : K_R \to \mathbb{C} - \{0; \infty\}$.

By comparing of (15) and (12) we deduce that

$$2g(z)f(z) = \frac{\tau_2 - \tau_1}{2\ln R}\cdot\frac{dz}{z}. \tag{16}$$

To exlude $\ln R$ from the last equality we substitute (16) into (12), and after using (10) we obtain

$$\ln R = \frac{\pi(\tau_2 - \tau_1)}{J_3}. \tag{17}$$

By substituting of the found relationship into (16) we arrive at the conclusion of Lemma 2.

4. Proof of Theorem 1

Let us denote $w = (J_1 + iJ_2)/J_3$. Combining Lemma 2, (12) and (9) we obtain

$$\int_{C_1} \frac{1-g^2(\zeta)}{2g(\zeta)}\frac{d\zeta}{\zeta} = 2\pi w_1 i, \qquad \int_{C_1} \frac{1+g^2(\zeta)}{2g(\zeta)}\frac{d\zeta}{\zeta} = 2\pi w_2.$$

Simplifying the last expressions and denoting $w = |w| \cdot e^{i\theta}$, $g_1(z) = -e^{-i\theta}g(z)$ we obtain the following system

$$\frac{1}{2\pi}\int_{C_1} \frac{g_1(\zeta)d\zeta}{\zeta} = |w|, \qquad \frac{1}{2\pi}\int_{C_1} \frac{d\zeta}{g_1(\zeta)\zeta} = -|w|.$$

Applying Theorem 2 we arrive at the inequality

$$\ln R \leq \frac{\pi^2}{|w| + \sqrt{1+|w|^2}}$$

where $|w| \equiv |J_1 + iJ_2|/|J_3| = \tan \alpha(\mathcal{M})$. Using (17) we obtain the required estimate and the theorem is proved.

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CZAS ŻYCIA MINIMALNYCH TUB I WSPÓŁCZYNNIKI FUNKCJI JEDNOLISTNYCH W PIERŚCIENIU KOŁOWYM

Streszczenie

Uzyskano oszacowanie czasu życia dwu-wymiarowych tub minimalnych w $\mathbb{R}^3$ przy użyciu metody z zakresu teorii potencjału oraz zbadano związek tego zagadnienia ze współczynnikami funkcji jednolistnych w pierścieniu.

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GENERALIZATIONS OF COMPLEX ANALYSIS AND THEIR APPLICATIONS IN PHYSICS II

Wiesław Majchrzak and Andrzej Szwankowski

THE BOUNDS OF SOME FUNCTIONAL FOR HOLOMORPHIC AND UNIVALENT FUNCTIONS WITH REAL COEFFICIENTS

Summary

In the paper the bounds of the functional $a_3^m(a_3 - \alpha a_2^2)$ in the well-known family S_R of functions holomorphic and univalent in the unit disc, with real coefficients, are determined for any $\alpha = \bar{\alpha}$ and $m = 1, 2, 3, \ldots$. In the proof, use is made of a suitable differential–functional equation for extremal functions.

1. Introduction

Let S denote the usual class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

holomorphic and univalent in the disc $\Delta = \{z : |z| < 1\}$, whereas $S_R \subset S$ — the subset consisting of all functions with real coefficients, i.e. $a_n = \bar{a}_n$, $n = 2, 3, \ldots$.

The class S_R is compact in the topology of locally uniform convergence, hence, for any real functional continuous in S_R, there exist, of course, functions called

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