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Subharmonicity of Higher Dimensional Exponential Transforms

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To Harold Shapiro on his 75th Anniversary, with admiration.

Abstract. Our main result states that the function $(1-E_\rho)^{(n-2)/n}$ is subharmonic, where $0 \le \rho \le 1$ is a density function in $\mathbb{R}^n$, $n \ge 3$, and $E_\rho(x) = \exp\left(-\frac{2}{n}\fint \frac{\rho(\zeta)d\zeta}{|\zeta-x|^n}\right)$, is the exponential transform of ρ. This answers in affirmative the recent question posed by B. Gustafsson and M. Putinar in [6].

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1. Introduction

The exponential transform can be viewed as a potential depending on a domain in $\mathbb{R}^n$, or, more generally, on a measure having a *density* function $\rho(x)$ (with compact support) in the range $0 \le \rho \le 1$. The two-dimensional version

$$E_\rho(z,w) = \exp\left[-\frac{1}{\pi}\int \frac{\rho(\zeta)\, dA(\zeta)}{(\zeta - z)(\bar{\zeta} - \bar{w})}\right] \tag{1.1}$$

has appeared in operator theory, as a determinantal-characteristic function of certain close to normal operators [4], [10], and has previously been studied and proved to be useful within operator theory, moment problems and other problems of domain identification, and for proving regularity of free boundaries (see [6], [11] for further references). A corresponding exponential transform on the real axis was already known and used by A.A. Markov (in the 19th century) and later by N.I. Akhiezer and M.G. Krein in their studies of one-dimensional moment problems [1], [2] (see, also [8]).

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In [6] the diagonal version of (1.1),

$$E_\rho(x) = \exp\left[-\frac{2}{\omega_n}\int \frac{\rho(\zeta)d\zeta}{|x-\zeta|^n}\right],$$

is studied in higher dimensional case $n \geq 3$. Here ω_n denotes the $(n-1)$-dimensional Lebesgue measure of the unit sphere in $\mathbb{R}^n$.

Clearly, $0 < E_\rho(x) < 1$ for all $x \notin \operatorname{supp}\rho$. In particular, it was shown in [6] that E_ρ is a subharmonic function. In two dimensions it is also known that the function $\ln(1-E_\rho)$ is subharmonic, which is a stronger statement. Here we extend the mentioned sub/superharmonicity in dimension $n \geq 3$, thereby answering in affirmative a recent question [6, p. 566]:

Theorem 1.1. *Let $E_\rho(x)$ be the exponential transform of a density $\rho \not\equiv 0$. Then the function*

$$\begin{cases} \ln(1-E_\rho), & \text{if} \quad n=2, \\ \frac{1}{n-2}(1-E_\rho)^{(n-2)/n}, & \text{if} \quad n \geq 3, \end{cases} \tag{1.2}$$

is subharmonic outside $\operatorname{supp}\rho$.

In fact, we show that a stronger version holds. To formulate it we need some notation. Given an integer $n \geq 1$, we define $\mathcal{M}_n(t)$ as the solution of the following ODE:

$$\mathcal{M}_n'(t) = 1 - \mathcal{M}_n^{2/n}(t), \qquad \mathcal{M}(0) = 0. \tag{1.3}$$

We call $\mathcal{M}_n(t)$ the *profile* function.

Theorem 1.2. *For $n \geq 2$ let ρ be a density function and*

$$V_\rho(x) = -\frac{n}{2}\ln E_\rho(x) \equiv \frac{n}{\omega_n}\int \frac{\rho(\zeta)d\zeta}{|x-\zeta|^n}. \tag{1.4}$$

Then the function

$$\begin{cases} \log \mathcal{M}_2(V_\rho(x)), & \text{if} \quad n=2 \\ [\mathcal{M}_n(V_\rho(x))]^{(n-2)/n}, & \text{if} \quad n \neq 2 \end{cases} \tag{1.5}$$

is subharmonic outside the support of ρ. Moreover, this function is harmonic in $\mathbb{R}^n \setminus B$, if B is an arbitrary Euclidean ball and $\rho = \chi_B$ is its characteristic function.

We discuss properties of the profile function in more detail in Section 4. In particular we show that $1 - \mathcal{M}_n(x)$ is a completely monotonic function in $\mathbb{R}^+$.

2. The main inequality

2.1. Variational problem

Let $x = (x_1, y) \in \mathbb{R}^n$, $y = (x_2, \ldots, x_n)$, and

$$\mathbb{R}^n_\pm = \{x = (x_1, y) : \pm x_1 > 0\}.$$

Given a measurable function $h(x)$ we denote by $\mathcal{J}(h)$ the integral

$$\mathcal{J}(h) = \fint_{\mathbb{R}^n} h(x)\, dx = \frac{n}{\omega_n} \int_{\mathbb{R}^n} h(x)\, dx$$

where $dx = dx_1 dy$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. In what follows we fix the following notations:

$$f(x) = \frac{1}{|x|^{n-2}}, \qquad g(x) = \frac{1}{|x|^n}, \qquad \varphi(x) = \frac{x_1}{|x|^n},$$

and suppose that $\rho(x)$ is a density function such that

$$0 \le \rho(x) \le 1.$$

If $n = 2$ we assume that $f(x) \equiv 1$. Throughout this section, unless otherwise stated, we will assume that $\rho \neq 0$ on a non-null set and the support of ρ does not contain a neighborhood of the origin. We write

$$\rho \in \mathcal{H}(w) \qquad \Leftrightarrow \qquad \mathcal{J}(\rho g) \equiv \fint_{\mathbb{R}^n} \frac{\rho}{|x|^n}\, dx = w \ge 0. \tag{2.1}$$

Our main subject is the ratio

$$\Phi(\rho) = \frac{\mathcal{J}^2(\varphi\rho)}{\mathcal{J}(f\rho)}.$$

Theorem 2.1. *Let ρ be a density function, $0 \notin \operatorname{supp}\rho$. Then*

$$\max_{\rho\in\mathcal{H}(w)} \Phi(\rho) = \mathcal{M}_n(w). \tag{2.2}$$

For any $w > 0$ the maximum is attained when $\rho(x)$ is the characteristic function of the ball centered at $(\tau, \mathbf{0})$ of radius $\tau\mathcal{M}_n(w)^{1/n}$, with $\tau > 0$.

We mention two limit cases of the last assertion. Namely, the boundedness of maximum in (2.2) easily follows from $\varphi^2 \le fg$ and the Cauchy-Schwarz inequality:

$$\frac{\mathcal{J}^2(\varphi\rho)}{\mathcal{J}(f\rho)} \le \mathcal{J}(g\rho) = w. \tag{2.3}$$

On the other hand, it was shown by Gustafsson and Putinar in [6, p. 563] that

$$\frac{\mathcal{J}^2(\varphi\rho)}{\mathcal{J}(f\rho)} < 1 \tag{2.4}$$

does hold. The last means that inequality (2.4) considerably refines (2.3) when $w > 1$ while the first estimate becomes to be sharper when w is a small value.

Corollary 2.2. *For any density function $\rho(x)$, $0 \notin \operatorname{supp}\rho$, the following sharp inequality holds*

$$\left(\fint_{\mathbb{R}^n} \frac{x_1\rho(x)}{|x|^n}\, dx\right)^2 \le \mathcal{M}_n \left(\fint_{\mathbb{R}^n} \frac{\rho(x)}{|x|^n}\, dx\right) \fint_{\mathbb{R}^n} \frac{\rho(x)}{|x|^{n-2}}\, dx \tag{2.5}$$

The inversion $x \to x/|x|^2$ gives another equivalent form of the preceding property.

Corollary 2.3. *For any density function $\rho(x)$, $0 \notin \operatorname{supp}\rho$, the following sharp inequality holds:*

$$\left(\fint_{\mathbb{R}^n} \frac{x_1\rho(x)}{|x|^{n+2}}\,dx\right)^2 \le \mathcal{M}_n\left(\fint_{\mathbb{R}^n} \frac{\rho(x)}{|x|^n}\,dx\right)\fint_{\mathbb{R}^n} \frac{\rho(x)}{|x|^{n+2}}\,dx \tag{2.6}$$

Remark 2.4. We note that for $n \ge 3$ the above inequality (2.5) can be interpreted as a pointwise estimate on the Coulomb potential

$$U_\rho(x) = \fint \frac{\rho(\zeta)dx}{|x-\zeta|^{n-2}}$$

with an *bounded* density function ρ, $0 \le \rho \le 1$. Indeed, using the inversion in $\mathbb{R}^n$ we see that (2.5) is equivalent to

$$|\nabla U_\rho(x)|^2 \le \mathcal{M}_n[V_\rho(x)]U_\rho(x), \qquad x \notin \operatorname{supp}\rho,$$

where $V_\rho(x)$ is defined by (1.4). In particularly, $\mathcal{M}_n(w) < 1$ gives us the inequality due to Gustafsson and Putinar [6]:

$$|\nabla U_\rho(x)|^2 < U_\rho(x), \qquad x \notin \operatorname{supp}\rho.$$

2.2. Auxiliary integrals

In order to prove Theorem 2.1, we need to evaluate the integrals in (2.5) for a specific choice of the density function. Namely, let $\tau > \alpha > 0$ and consider the following density function

$$\widehat{\rho}(x) = \chi_{\mathbb{D}}(x),$$

where

$$\mathbb{D} \equiv \mathbb{D}(\alpha,\tau) := \Big\{x = (x_1, y): \quad (x_1-\tau)^2 + |y|^2 < \tau^2 - \alpha^2\Big\}. \tag{2.7}$$

First, we note that the function $f(x) = |x|^{2-n}$ is harmonic in $\overline{\mathbb{D}}$. Using the fact that the ball $\mathbb{D}$ is of radius $\sqrt{\tau^2-\alpha^2}$ and centered at $x = (\tau, \mathbf{0})$, we have by the mean value theorem

$$\mathcal{J}(\widetilde{\rho} f) = \fint_{\mathbb{D}} \frac{dx}{|x|^{n-2}} = \frac{(\tau^2-\alpha^2)^{n/2}}{\tau^{n-2}} = \alpha^2 \frac{\sinh^n \xi}{\cosh^{n-2}\xi} \tag{2.8}$$

where

$$\cosh\xi = \frac{\tau}{\alpha}. \tag{2.9}$$

Similarly, harmonicity of $\varphi(x) = x_1|x|^{-n}$ implies

$$\mathcal{J}(\widetilde{\rho}\varphi) = \alpha\frac{\sinh^n\xi}{\cosh^{n-1}\xi}.$$

To evaluate $\mathcal{J}(\widetilde{\rho} g)$ we consider the following auxiliary function

$$\lambda(x) = \frac{|x|^2+\alpha^2}{2\tau x_1}.$$

Then $\lambda(x)$ is positive on $\mathbb{D}$ and ranges in

$$\frac{\alpha}{\tau} \le \lambda(x) < 1, \quad x \in \mathbb{D}.$$

Moreover, it is easy to see that

$$\lambda(x) \equiv z, \qquad x \in S(z) = \partial\mathbb{D}(\alpha, \tau z). \tag{2.10}$$

Hence, the co-area formula yields

$$\mathcal{J}(\widetilde{\rho} g) = \fint_{\mathbb{D}} \frac{dx}{|x|^n} = \frac{n}{\omega_n} \int_{\alpha/\tau}^{1} dz \int_{S(z)} \frac{dS}{|x|^n |\nabla\lambda(x)|}. \tag{2.11}$$

Here dS is the $(n-1)$-dimensional surface measure of the level set $S(z)$.

On the other hand, we have for the gradient

$$|\nabla\lambda|^2 = \frac{|y|^2}{\tau^2 x_1^2} + \frac{(x_1^2 - \alpha^2 - |y|^2)^2}{4\tau^2 x_1^4},$$

which by virtue of (2.10) implies the corresponding value on the level set $S(z)$:

$$|\nabla\lambda|^2 \Big|_{S(z)} = \frac{\tau^2 z^2 - \alpha^2}{\tau^2 x_1^2}.$$

Substitution of the last expression into (2.11) yields

$$\mathcal{J}(\widetilde{\rho} g) = \frac{n}{\omega_n} \int_{\alpha/\tau}^{1} \frac{\tau dz}{\sqrt{\tau^2 z^2 - \alpha^2}} \int_{E(z)} \frac{x_1}{|x|^n} dS.$$

Since $\varphi(x) = x_1 |x|^{-n}$ in the inner integral is a harmonic function and $S(z)$ is a sphere, we have by the mean value theorem

$$\mathcal{J}(\widetilde{\rho} g) = \frac{n}{\omega_n} \int_{\alpha/\tau}^{1} \frac{\tau dz}{\sqrt{\tau^2 z^2 - \alpha^2}} \cdot \frac{(\tau^2 z^2 - \alpha^2)^{(n-1)/2}}{(\tau z)^{n-1}} = n \int_0^{\xi} \tanh^{n-1} t dt.$$

where ξ is defined by (2.9). Thus we obtain

$$\mathcal{J}(\widetilde{\rho} g) = T_n(\tau/\alpha) = T_n(\xi) := n \int_0^{\xi} \tanh^{n-1} t\, dt. \tag{2.12}$$

We point out that *the latter integral depends only on the ratio* τ/α. One can easy verify that

$$\mathcal{M}_n(T_n(\xi)) \equiv \tanh^n \xi = \left(\frac{\sqrt{\tau^2 - \alpha^2}}{\tau} \right)^n. \tag{2.13}$$

Remark 2.5. After a suitable shift in the x_1-direction, the last computation is equivalent to the following relation

$$\mathcal{M}_n \left(\fint_{\mathbb{B}(R)} \frac{d\zeta}{|x - \zeta|^n} \right) = \left(\frac{R}{|x|} \right)^n, \tag{2.14}$$

which holds for any ball $\mathbb{B}(R)$ of radius R centered at the origin.

2.3. Proof of Theorem 2.1

Let us denote

$$M_n^+(w) = \sup_{\rho\in\mathcal{R}} \Phi(\rho) \tag{2.15}$$

where $\mathcal{R}$ denotes the class of all density functions ρ such that $\operatorname{supp}\rho\cap\mathbb{R}^n_-$ has null measure. Then Theorem 2.1 follows from the following lemmas.

Lemma 2.6. $M_n^+(w) = \mathcal{M}_n(w)$.

Lemma 2.7. $\sup_\rho \Phi(\rho) = M_n^+(w)$.

Proof of Lemma 2.6. Our first step is to reduce the problem (2.15) to the following linear extremal problem with additional constraints:

$$N_n(w) := \sup_{\rho\in\mathcal{R}}\{\mathcal{J}(\rho\varphi):\quad \mathcal{J}(\rho f)=1,\ \mathcal{J}(\rho g)=w\}. \tag{2.16}$$

Then we have

$$M_n^+(w) = N_n^2(w). \tag{2.17}$$

Indeed, in order to prove (2.17), let $\rho_a(x)=\rho(ax)$ be a homothety of $\rho(x)$ with positive coefficient a. Clearly, this transformation preserves the class $\mathcal{R}$. On the other hand, one can easily see that

$$\Phi(\rho_a) = \Phi(\rho)$$

by the virtue of homogeneity of Φ. Moreover,

$$\mathcal{J}(\rho_a\varphi) = \frac{1}{a}\mathcal{J}(\rho\varphi), \qquad \mathcal{J}(\rho_a f) = \frac{1}{a^2}\mathcal{J}(\rho f),$$

which proves (2.17).

Next, we claim that for any nonnegative w there exists an $\alpha>0$ and $\tau>\alpha$ such that

$$\mathcal{J}(\widetilde{\rho} f)=1, \qquad \mathcal{J}(\widetilde{\rho} g)=w, \tag{2.18}$$

where $\widetilde{\rho}=\chi_{\mathbb{D}(\alpha,\tau)}$ is the characteristic function of the ball $\mathbb{D}(\alpha,\tau)$ in (2.7). Indeed, using the definition of function $T_n(t)$ in (2.12) one can easily see that there exist a unique root $\xi>0$ of the equation

$$T_n(\xi)=w. \tag{2.19}$$

Then we chose $\alpha>0$ such that

$$\alpha^2 = \frac{\cosh^{n-2}\xi}{\sinh^n\xi},$$

and let $\tau=\alpha\cosh\xi$. Now (2.18) immediately follows from (2.8) and (2.12).

Thus, the function $\widetilde{\rho}(x)$ satisfies (2.18) and it follows that it is admissible for the problem (2.16). This implies

$$N_n(w) \ge \mathcal{J}(\widetilde{\rho}\varphi).$$

To prove that the inverse inequality holds, we fix any function $\rho \in \mathcal{R}$ which is admissible for (2.16). Then

$$\mathcal{J}(\widetilde{\rho}(f+\alpha^2 g)) = \mathcal{J}(\rho(f+\alpha^2 g)) = 1+\alpha^2 w.$$

The last property means that both the functions ρ and $\widetilde{\rho}$ are test functions for the following extremal problem

$$\sup_{\rho\in\mathcal{R}}\{\mathcal{J}(\rho\varphi): \quad \mathcal{J}(\rho(f+\alpha^2 g)) = 1+w\alpha^2\}. \tag{2.20}$$

Let us consider the ratio

$$h(x) := \frac{\varphi(x)}{f(x)+\alpha^2 g(x)} = \frac{x_1}{|x|^2+\alpha^2}.$$

Then,

$$\{x\in\mathbb{R}^n : h(x) > \frac{1}{2\tau}\} = \mathbb{D}(\alpha,\tau),$$

and it follows from the Bathtub Principle [9, p. 28] that $\widetilde{\rho}$ is the extremal density for (2.20). Thus, we have

$$\mathcal{J}(\rho\varphi) \le \mathcal{J}(\widetilde{\rho}\varphi),$$

and consequently

$$N_n(w) \le \mathcal{J}(\widetilde{\rho}\varphi).$$

Hence, we conclude that

$$N_n(w) = \mathcal{J}(\widetilde{\rho}\varphi) = \alpha\frac{\sinh^n\xi}{\cosh^{n-1}\xi}.$$

Now, it follows from (2.17) and our choice of α that

$$M_n^+(w) = N_n^2(w) = \alpha^2\frac{\sinh^{2n}\xi}{\cosh^{2n-2}\xi} = \tanh^n\xi,$$

and from (2.13), we find

$$M_n^+(w) = \mathcal{M}_n(T_n(\xi)) = \mathcal{M}_n(w),$$

and the lemma follows. □

Proof of Lemma 2.7. It suffices only to prove the one-side inequality

$$\sup_\rho \Phi(\rho) \le M_n^+(w). \tag{2.21}$$

Let ρ is an arbitrary admissible for (2.1) density function. Excluding the trivial case $\rho\in\mathcal{R}$ we distinguish two rest cases:

(i) the set supp $\rho\cap\mathbb{R}^n_+$ has the null measure;

(ii) the set supp ρ has non-zero counterpart in the both half-spaces.

Let ρ satisfy (i). Then the function

$$\rho^*(x_1,y) := \rho(-x_1,y)$$

belongs to $\mathcal{R}$, and it follows that

$$\sup_{\rho\in(\mathrm{i})} \Phi(\rho) = \sup_{\rho\in\mathcal{R}} \Phi(\rho) = M_n^+(w). \tag{2.22}$$

Now, let ρ satisfy (ii). We set $\rho^{\pm}(x) = \chi_{\mathbb{R}^n_\pm}(x)\rho(x)$. Then

$$\mathcal{J}(\rho\varphi) = \mathcal{J}(\rho^+\varphi) - \mathcal{J}((\rho^-)^*\varphi),$$
$$\mathcal{J}(\rho f) = \mathcal{J}(\rho^+ f) + \mathcal{J}((\rho^-)^* f),$$

where the last integrals are positive. Using the elementary inequality

$$\frac{(a-b)^2}{c+d} \le \max\left[\frac{a^2}{c}, \frac{b^2}{d}\right]$$

which holds for any set of positive numbers a, b, c, d, we conclude that

$$\Phi(\rho) = \frac{\mathcal{J}^2(\rho\varphi)}{\mathcal{J}(\rho f)} \le \max[\Phi(\rho^+), \Phi((\rho^-)^*)].$$

Hence, we have by Lemma 2.6

$$\Phi(\rho) \le \max[M_n^+(w_1), M_n^+(w_2)] = \max[\mathcal{M}_n(w_1), \mathcal{M}_n(w_2)],$$

where

$$w_1 = \mathcal{J}(\rho^+ g), \qquad w_2 = \mathcal{J}((\rho^+)^* g).$$

But

$$w = \mathcal{J}(\rho g) = w_1 + w_2,$$

whence $w_i \le w$, $i = 1, 2$. Since $\mathcal{M}_n$ is an increasing function we obtain $\Phi(\rho) \le \mathcal{M}_n(w)$, and consequently

$$\sup_{\rho\in(\mathrm{ii})} \Phi(\rho) \le \mathcal{M}_n(w) = M_n^+(w).$$

Combining the last inequality with (2.22) we obtain

$$\sup_{\rho} \Phi(\rho) = \sup_{\mathcal{R}\cup(\mathrm{i})\cup(\mathrm{ii})} \Phi(\rho) \le M_n^+(w)$$

which proves (2.21). □

3. Proof of the main results

Lemma 3.1. *For any $n \ge 1$ we have*

$$\mathcal{M}_n(w) \le Q_n(w) := \frac{e^{2w/n} - 1}{e^{2w/n} - \frac{n-2}{n}}. \tag{3.1}$$

Proof. Note that in the cases $n = 1, 2$, we have

$$\mathcal{M}_1(w) = \tanh w = \frac{e^{2w} - 1}{e^{2w} + 1},$$
$$\mathcal{M}_2(w) = 1 - e^{-w}$$

which turns (3.1) into equality.

Now, let $n \geq 3$. We have $M_n(0) = Q_n(0) = 0$ and by Definition (1.3) it suffices only to prove that

$$Q_n'(w) \geq 1 - Q_n^{2/n}, \quad w > 0. \tag{3.2}$$

We have

$$Q_n'(w) = (1 - \frac{n-2}{n} Q_n)(1 - Q_n)$$

and (3.2) becomes to be equivalent to the inequality

$$\frac{1 - t^{1-\gamma}}{1 - t} < 1 - \gamma t,$$

where $t = Q_n(w) \in (0, 1)$ and $\gamma = (n - 2)/n$. To verify the last inequality we rewrite it in the form

$$\frac{1 - t^\gamma}{1 - t} > \gamma t^\gamma.$$

For $t \in (0, 1)$, the function in the left hand side is a decreasing function while the right hand side member is an increasing one. Since the both functions have the same limit value γ at $t = 1$, we have the desired inequality. □

Proof of Theorem 1.1. Let $f(x)$ denote the function in (1.2). Then we have for any $n \geq 2$ and $x \notin \operatorname{supp} \rho$

$$\nabla f(x) = -(1 - E_\rho)^{-2/n} \nabla E_\rho,$$
$$\Delta f(x) = -\frac{2}{n}(1 - E_\rho)^{-\frac{2+n}{n}} \left[\frac{n}{2}(1 - E_\rho)\Delta E_\rho + |\nabla E_\rho|^2 \right].$$

Then the inequality $\Delta f(x) \geq 0$ to be proved becomes

$$\frac{n}{2}(1 - E_\rho)\Delta E_\rho + |\nabla E_\rho|^2 \leq 0. \tag{3.3}$$

On the other hand,

$$\nabla E_\rho(x) = 2E_\rho(x) \fint \frac{(x - \zeta)\rho(\zeta) d\zeta}{|x - \zeta|^{n+2}},$$
$$\Delta E_\rho(x) = 4E_\rho(x) \left(\left| \fint \frac{(x - \zeta)\rho(\zeta) d\zeta}{|x - \zeta|^{n+2}} \right|^2 - \fint \frac{\rho(\zeta) d\zeta}{|x - \zeta|^{n+2}} \right),$$

and (3.3) becomes

$$\left(1 - \frac{n-2}{n} E_\rho \right) \left| \fint \frac{(x - \zeta)\rho(\zeta) d\zeta}{|x - \zeta|^{n+2}} \right|^2 \leq (1 - E_\rho) \fint \frac{\rho(\zeta) d\zeta}{|x - \zeta|^{n+2}}. \tag{3.4}$$

In order to prove (3.4) we can assume without loss of generality that $x = 0$. In this case, after a suitable rotation we can write the vector integral as follows

$$\left|\fint \frac{\zeta\rho(\zeta)d\zeta}{|\zeta|^{n+2}}\right| = \fint \frac{\zeta_1\rho(\zeta)d\zeta}{|\zeta|^{n+2}}.$$

Thus, we arrive at the inequality to be proved

$$\left(1 - \frac{n-2}{n}e^{-\frac{2w}{n}}\right)\left(\fint \frac{\zeta_1\rho(\zeta)d\zeta}{|\zeta|^{n+2}}\right)^2 \leq (1 - e^{-\frac{2w}{n}})\fint \frac{\rho(\zeta)d\zeta}{|\zeta|^{n+2}}, \tag{3.5}$$

with

$$w = \fint \frac{\rho(\zeta)d\zeta}{|\zeta|^{n}}.$$

But, it is easy to see that (3.5) follows from Corollary 2.3 and Lemma 3.1. The theorem follows. □

Proof of Theorem 1.2. Let $F(x)$ denote the function in (1.5) and $V(x) = V_\rho(x)$. Then the argument similar to that above yields for $n \geq 3$

$$\begin{aligned}\Delta F(x) &= \Delta(\mathcal{M}_n(V))^{(n-2)/n}\\ &= (1 - \mathcal{M}_n^{2/n}(V))\left(\frac{n-2}{n}\mathcal{M}_n^{-\frac{2}{n}}(V)\Delta V - \frac{2(n-2)}{n^2}\mathcal{M}_n^{-\frac{2+n}{n}}(V)|\nabla V|^2\right)\\ &= 2(n-2)(1 - \mathcal{M}_n^{2/n}(V))\Big[\mathcal{M}_n(V)B - |A|^2\Big],\end{aligned} \tag{3.6}$$

where

$$A = \fint \frac{(x-\zeta)\rho(\zeta)d\zeta}{|x-\zeta|^{n+2}}, \qquad B = \fint \frac{\rho(\zeta)d\zeta}{|x-\zeta|^{n+2}}.$$

Similarly, we have for $n = 2$

$$\Delta F(x) = \frac{1 - \mathcal{M}_2(V)}{\mathcal{M}_2^2(V)}\Big[\mathcal{M}_2(V)B - |A|^2\Big].$$

Hence, for all integer $n \geq 2$, the sign of the Laplacian $\Delta F(x)$ coincides with the sign of $[M_n(V)B - |A|^2]$.

Let us fix an arbitrary point $x \notin \operatorname{supp}\rho$. Then after a suitable rotation we can reduce the vector integral A to the scalar one such that the value in last brackets in (3.6) becomes

$$\mathcal{M}_n\left(\fint \frac{\rho_1(\zeta)d\zeta}{|\zeta|^n}\right)\fint \frac{\rho_1(\zeta)d\zeta}{|\zeta|^{n+2}} - \left(\fint \frac{\zeta_1\rho_1(\zeta)d\zeta}{|\zeta|^{n+2}}\right)^2,$$

where $\rho_1(\zeta)$ is the correspondent transformed density. Then Corollary 2.3 again implies that the latter difference is nonnegative and subharmonicity of $\mathcal{E}_\rho$ easily follows.

Now, let us prove the second assertion of the theorem. Let $\mathbb{B}(R)$ be the ball of radius R with center at the origin and $\widehat{\rho}(x) = \chi_{\mathbb{B}(R)}(x)$ be the corresponding characteristic function. Then

$$V_{\widehat{\rho}}(x) := \int_{\mathbb{B}(R)} \frac{d\zeta}{|x-\zeta|^n},$$

and we have from (2.14) that in this case

$$\mathcal{M}_n(V_{\widehat{\rho}}(x)) = \left(\frac{R}{|x|}\right)^n,$$

which obviously yields harmonicity of

$$[\mathcal{M}_n(V_{\widehat{\rho}}(x))]^{(n-2)/n} = R^{n-2}|x|^{2-n}$$

for $n \geq 3$, and

$$\ln \mathcal{M}_2(V_{\widehat{\rho}}(x)) = 2\ln \frac{R}{|x|},$$

if $n = 2$. The theorem is completely proved. □

4. The profile function

Here we study the profile function $\mathcal{M}_n$ in more detail. This higher transcendental function, apart of its appearance in the above theorems, admits also number-theoretical applications (e.g., in connection with the Euler-Mascheroni constant γ, see Section 4.2). Our main result (Theorem 4.1 below) states that $1 - M_n(w)$ is a completely monotonic function. We also show (Theorem 4.5) that this function can be analytically extended across $w = +\infty$ by making use of a specific logarithmic transformation.

4.1. Complete monotonicity

It is convenient to consider the general case of (1.3). Namely, given a real $\alpha > 0$ we define $F_\alpha(x)$ as a solution to the following ODE

$$F'_\alpha(x) = 1 - F^\alpha_\alpha(x), \qquad F_\alpha(0) = 0. \tag{4.1}$$

Then for an integer n we have $\mathcal{M}_n(w) = F_{2/n}(w)$.

We recall that a function $f(x)$ defined on $[0;+\infty)$ is said to be *completely monotonic* if

$$(-1)^k f^{(k)}(x) \geq 0, \qquad x \in \mathbb{R}^+.$$

Theorem 4.1. *Let* $\alpha > 0$. *Then*

(i) $F_\alpha(x)$ *is an increasing function for* $x \geq 0$ *such that* $F_\alpha(x) : \mathbb{R}^+ \to [0;1)$;
(ii) *for all* $\alpha \in (0;1]$ *the function*

$$\widetilde{F}_\alpha(x) = 1 - F_\alpha(x)$$

is completely monotonic on $\mathbb{R}^+$.

It follows from the well-known Bernstein theorem [3] (see also [13, p. 161]) that $\widetilde{F}_\alpha(x)$ is a Laplace transform of a positive measure supported on $\mathbb{R}^+$.

Corollary 4.2. *For all $\alpha \in (0;1]$ the following Laplace-Stieltjes representation holds:*

$$\widetilde{F}_\alpha(x) = \int_0^{+\infty} e^{-xt}\, d\sigma_\alpha(t) \tag{4.2}$$

where $d\sigma_\alpha$ is a positive probability measure with finite variation,

$$\int_0^{+\infty} d\sigma_\alpha(t) = \widetilde{F}_\alpha(0) = 1. \tag{4.3}$$

The following subadditive property is a consequence of the general result due to Kimberling [7] and concerns complete monotonic functions satisfying (4.3).

Corollary 4.3. *For all $0 < \alpha \le 1$ the function $\widetilde{F}_\alpha(x)$ is subadditive in the sense that*

$$\widetilde{F}_\alpha(x)\widetilde{F}_\alpha(y) \le \widetilde{F}_\alpha(x+y). \tag{4.4}$$

Remark 4.4. It is easy to verify that for $\alpha > 1$ the third derivative of $F_\alpha'''(x)$ has no constant sign on $\mathbb{R}^+$. Thus our constraint is optimal for positive values of α. On the other hand, if $\alpha = 1$ then $F_1(x)$ can be derived as follows

$$F_1(x) = 1 - e^{-x}, \qquad \widetilde{F}_1(x) = e^{-x}$$

which implies the complete monotonicity immediately. Moreover, in the latter case $\widetilde{F}_1(x)$ satisfies a full additive property instead of (4.4). We notice also that in this case one can easily find that $d\sigma_1(t) = \delta_1(t)$ the delta-Dirac probability measure supported at $t = 1$. More precisely, we have

$$\sigma_1(t) = \chi_{[1,+\infty)}(t).$$

Proof of Theorem 4.1. The only non-trivial part of the theorem is (ii). We notice first that

$$\widetilde{F}_\alpha(x) \ge 0, \qquad \widetilde{F}_\alpha^{(k)}(x) = -F_\alpha^{(k)}(x), \quad k = 1, 2, \ldots$$

and

$$F_\alpha''(x) = -\alpha(1 - F_\alpha^\alpha)\frac{F_\alpha^\alpha}{F_\alpha(x)}.$$

On the other hand, one can easily show by induction that the following property holds for all $k \ge 0$

$$F_\alpha^{(k+2)}(x) = \alpha t(1-t)\frac{H_k(t)}{F_\alpha(x)^{k+1}} \tag{4.5}$$

where

$$t = F_\alpha^\alpha(x)$$

and $H_j(t)$ is a polynomial of degree at most j. Moreover, we have the following recurrent relationship

$$H_{k+1}(t) = [(k+1-2\alpha)t - (k+1-\alpha)]H_k(t) + \alpha t(1-t)H_k'(t), \qquad k \geq 2 \quad (4.6)$$

with initial condition

$$H_0(t) = -1. \quad (4.7)$$

Since $t = F_\alpha^\alpha(x)$ ranges in $[0;1)$ we have only to prove that the polynomials $(-1)^{k+1}H_k(t)$ are nonnegative in $\Delta = [0,1)$.

We will use the following Bernstein-type transformation

$$P^*(z) = (1+z)^n P\left(\frac{1}{1+z}\right), \qquad n \geq \deg P$$

which transforms a polynomial P to a polynomial of degree at most n.

Let

$$P(t) = a_0 + a_1 t + \ldots + a_n t^n$$

(here we use the assumption that $\deg P \leq n$ and some coefficients may vanish). Then we can write

$$P(t) = \sum_{j=0}^{n} b_j t^{n-j}(1-t)^j \quad (4.8)$$

where

$$P^*(z) = b_0 + b_1 z + \ldots + b_n z^n, \qquad z = \frac{1-t}{t}.$$

We recall that (4.8) is the Bernstein-type expansion of P by the basis $t^j(1-t)^{n-j}$.

It follows then from (4.8) that if all (non-zero) coefficients of the associate polynomial $P^*(z)$ have the same sign: $\operatorname{sgn} b_j = \varepsilon$, then $P(t)$ changes no sign in Δ and its sign coincides with ε.

Let $H_k^*(z)$ be the associative polynomial for $H_k(t)$. Then

$$H_k(t) = t^k H_k^*\left(\frac{1-t}{t}\right)$$

and

$$H_k'(t) = kt^{k-1}H_k^*\left(\frac{1-t}{t}\right) - t^{k-2}H_k^{*\prime}\left(\frac{1-t}{t}\right).$$

It follows from (4.6) that

$$-H_{k+1}^*(z) = [\alpha + (k+1)(1-\alpha)z]H_k^*(z) + \alpha z(1+z)H_k^{*\prime}(z). \quad (4.9)$$

We notice that by (4.7)

$$H_0^* = H_0 = -1.$$

On the other hand, since $0 \leq \alpha \leq 1$ the multipliers $(\alpha+(k+1)(1-\alpha)z)$ and $\alpha z(1+z)$ in (4.9) have positive coefficients with respect to z. Hence, it immediately follows from (4.9) by induction that all coefficients of $-H_{k+1}^*(z)$ have the same sign as

$H_k^*(z)$ does. Moreover, the sign of the coefficients of $H_k^*(z)$ is $(-1)^{k+1}$ which yields by the above remark that

$$(-1)^{k+1} H_k(t) \ge 0, \qquad t \in \Delta.$$

Clearly, the last property together with (4.5) yields the desired assertion. □

4.2. Exponential series for the profile function

Here we establish an explicit form of the above exponential representation for $\mathcal{M}_n(x)$. As above, it is convenient to consider a general $F_\alpha(x)$ instead of $\mathcal{M}_n(x)$ (see the definition (4.1)).

Let

$$\phi_\alpha(t) := 1 - F_\alpha\left(-\frac{1}{\alpha}\ln t\right).$$

According to its definition, $\phi_\alpha(t)$ is defined in $(0,1]$. But it turns out that a stronger property holds

Theorem 4.5. *The following properties hold:*

(i) *For any $\alpha > 0$ the function $\phi_\alpha(t)$ admits an analytic continuation on $(-\epsilon, 1)$ with some $\epsilon > 0$ depending on α.*

(ii) *The corresponding Taylor series at $t = 0$ are*

$$\phi_\alpha(t) = \sum_{k=1}^{\infty} \sigma_k (\gamma_\alpha t)^k, \tag{4.10}$$

where

$$\gamma(\alpha) = \frac{1}{\alpha}\exp\Bigl(-\int_0^1 \frac{1 - x^{\frac{1-\alpha}{\alpha}}}{1-x}\, dx\Bigr),$$

and σ_k are the coefficients defined by the following recurrence

$$\sigma_1 := 1, \quad \sigma_k = \frac{1}{k(k-1)} \sum_{\nu=1}^{k-1} \sigma_\nu \sigma_{k-\nu} [(1+\alpha)\nu - \alpha k]\nu. \tag{4.11}$$

(iii) *If $\alpha \in (0,1)$ then $\sigma_k > 0$ for all $k \ge 1$ and series* (4.10) *converges in* $(-1,1)$.

(iv) *For all $0 < \alpha < 1$, $\phi_\alpha(t)$ is a strictly increasing convex function in* $(-\infty, 1)$.

Remark 4.6. The exact value of γ_α has the following form

$$\ln \gamma_\alpha = -\Psi(1/\alpha) - \gamma + \ln(1/\alpha), \tag{4.12}$$

where $\Psi(z)$ is the Digamma function: $\Psi(z) = \Gamma'(z)/\Gamma(z)$, and $\gamma = 0.5772156\ldots$ is the Euler-Mascheroni constant. The assertion of the theorem is still valid for $\alpha = 0$ which formally corresponds to $n = \infty$. In this case, $\phi_0(x)$ satisfies the following ODE:

$$\phi_0'(x) = -\frac{\ln(1 - \phi_0(x))}{x}, \qquad \phi_0(0) = 0.$$

It follows from (4.12) that in this case $\gamma_0 = e^\gamma$.

Corollary 4.7. *Let $n \geq 2$ be an integer. Then*

$$1 - \mathcal{M}_n(x) = \sum_{k=1}^{\infty} a_k e^{-2kx/n},$$

where $a_k = \sigma_k \gamma_{2/n}^k > 0$ and the series converges for all $x \geq 0$. In particular, the measure in (4.2) *is an atomic measure supported at the set $\frac{2}{n}\mathbb{Z}^+$.*

We are grateful to Björn Gustafsson for pointing out another useful consequence of the preceding property. Let us define an (n-dimensional) version of the exponential transform as follows

$$\mathbb{E}_\rho(x) = 1 - \mathcal{M}_n\left(\frac{n}{\omega_n} \int \frac{\rho(\zeta)d\zeta}{|x-\zeta|^n}\right).$$

where ρ is a density function.

Corollary 4.8. *The function $E_\rho(x)$ is analytic if and only $\mathbb{E}_\rho(x)$ is. Moreover, these functions are linked by the following identity*

$$\mathbb{E}_\rho(x) = \phi_{2/n} \circ E_\rho(x). \tag{4.13}$$

Proof. The cases $n = 1$ and $n = 2$ are trivial. For $n \geq 3$ we notice that the desired property follows from (4.13) and the fact that $\phi'_\alpha(0) \neq 0$ (see (4.18) below). □

Proof of Theorem 4.5. First we consider (i). The case $\alpha = 1$ is trivial. Let $\alpha > 0$, $\alpha \neq 1$ and $F_\alpha(x)$ be the solution to (4.1). We notice that this function is determined uniquely by virtue of the condition $F_\alpha(0) = 0$, and it is a real analytic function of x in $(0, +\infty)$. It follows that $\phi_\alpha(t)$ also is a real analytic function of t for $t \in (0, 1)$ it is bounded there: $|\phi_\alpha(t)| < 1$. Moreover, $y = \phi_\alpha(t)$ satisfies the following differential equation

$$y'(t) = \frac{1 - (1 - y(t))^\alpha}{\alpha t}, \quad t \in (0, 1), \tag{4.14}$$

and the initial condition has to be transformed to $\phi_\alpha(1) = 0$.

Now we prove that $\phi_\alpha(x)$ admits an analytic continuation in a small disk in the complex plane. Let us define the following auxiliary function

$$S(\zeta) := \exp\left(-\int_\zeta^1 \frac{\alpha d\xi}{1 - (1 - \xi)^\alpha}\right).$$

Here we fix the branch of $(1 - \xi)^\alpha$ which assumes the value 1 at $\xi = 0$. Then $S(\zeta)$ is a single-valued holomorphic function in the unit disk $\mathbb{D}(1)$, where

$$\mathbb{D}(r) = \{\xi \in \mathbb{C} : |\zeta| < r\}.$$

Moreover, we have $S(\zeta) \neq 0$ and

$$S'(\zeta) = \frac{\alpha S(\zeta)}{1 - (1 - \zeta)^\alpha}. \tag{4.15}$$

On the other hand, we notice that the following renormalization of the above integrand

$$\frac{\alpha}{1-(1-\xi)^{\alpha}}-\frac{1}{\xi}=\frac{\alpha\xi-1+(1-\xi)^{\alpha}}{(1-(1-\xi)^{\alpha})\xi}=\frac{\alpha-1}{2}+\frac{(\alpha-1)(2\alpha-1)}{12}\xi+\dots$$

is an analytic function in $\mathbb{D}$ since it admits a regular Taylor expansion near $\xi=0$. This allows us to rewrite the above definition of $S(\zeta)$ as follows:

$$S(\zeta)=\zeta\,\exp\Bigl\{-\int_{\zeta}^{1}\Bigl(\frac{\alpha}{1-(1-\xi)^{\alpha}}-\frac{1}{\xi}\Bigr)d\xi\Bigr\}.$$

In particular, this implies

$$c_{\alpha}:=S'(0)=\exp\Bigl(-\int_{0}^{1}\Bigl(\frac{\alpha}{1-(1-x)^{\alpha}}-\frac{1}{x}\Bigr)dx\Bigr)\neq 0. \tag{4.16}$$

On the other hand,

$$\operatorname{Re}\frac{S'(\zeta)\zeta}{S(\zeta)}=\operatorname{Re}\frac{\zeta}{1-(1-\zeta)^{\alpha}}=\operatorname{Re}\frac{1}{\alpha}(1+\frac{\alpha-1}{2}\zeta+\dots).$$

Hence for $r>0$ sufficiently small,

$$\operatorname{Re}\frac{S'(\zeta)\zeta}{S(\zeta)}>0,\quad \zeta\in\mathbb{D}(r) \tag{4.17}$$

Taking into account (4.16), (4.17), and the well-known Alexander's property [5, p. 41] we conclude that the function $S(z)$ is starlike in $\mathbb{D}(r)$, and therefore univalent there.

Let $\psi(z)$ be the inverse function to $S(z)$. Clearly, it is defined in some small disk $\mathbb{D}(\epsilon)$ which is contained in the image $S(\mathbb{D}(r))$. Moreover, by its definition $\psi(z)$ assumes real values for real $z\in\mathbb{D}(\epsilon)$. We also have

$$\psi(0)=0,\qquad \psi'(0)=\frac{1}{S'(0)}=\frac{1}{c_{\alpha}}. \tag{4.18}$$

Furthermore, differentiation of the identity $S(\psi(z))=z$ together with (4.15) yields

$$1=S'(\psi(z))\psi'(z)=\frac{z\psi'(z)}{1-(1-\psi(z))^{\alpha}},$$

consequently, $y=\psi(z)$ is a solution of (4.14) in $\mathbb{D}(\epsilon)$.

Our next step is to prove that $\psi(z)$ is the desired analytic continuation. One suffices to show that $\psi(x)=\phi_{\alpha}(x)$ in some open subinterval of $(0,1)$, that in turn, is equivalent to establishing of the following identity

$$F_{\alpha}(x)=1-\psi(e^{-\alpha x}) \tag{4.19}$$

for all x in some interval $\Delta\subset(0,+\infty)$.

Taking into account the above remarks, we note that $g(x):=1-\psi(e^{-\alpha x})$ is a real-valued solution of (4.1) in $\Delta:=(-\frac{1}{\alpha}\ln\epsilon,+\infty)$. On the other hand, since $g(x)$

satisfies an obvious inequality $g(x) < 1$, and by virtue of the autonomic character of (4.1), we conclude that

$$g(x) = F_\alpha(x + c), \quad x \in \Delta,$$

for some constant $c \in \mathbb{R}$. Thus, we have only to check that $c = 0$.

To this aim, we note that

$$\alpha x = \int_0^{F_\alpha(x)} \frac{\alpha dt}{1 - t^\alpha} = -\int_0^{F_\alpha^\alpha(x)} \frac{1 - \tau^{\frac{1-\alpha}{\alpha}}}{1 - \tau} d\tau + \int_0^{F_\alpha^\alpha(x)} \frac{d\tau}{1 - \tau} d\tau.$$

Since $\lim_{\tau\to\infty} F_\alpha(\tau) = 1$, we arrive at

$$\lim_{x\to+\infty} (1 - F_\alpha^\alpha(x))e^{x\alpha} = \exp\Big(-\int_0^1 \frac{1 - \tau^{\frac{1-\alpha}{\alpha}}}{1 - \tau}\, d\tau\Big) = \alpha\gamma_\alpha,$$

or

$$\lim_{x\to+\infty} (1 - F_\alpha(x))e^{x\alpha} = \gamma_\alpha.$$

As a consequence we have,

$$\begin{aligned} \lim_{x\to+\infty} \psi(e^{-x\alpha})e^{x\alpha} &= \lim_{x\to+\infty} (1 - g(x))e^{x\alpha} \\ &= \lim_{x\to+\infty} [1 - F_\alpha(x + c)]e^{x\alpha} = e^{-c\alpha}\gamma_\alpha. \end{aligned} \tag{4.20}$$

On the other hand,

$$\lim_{x\to+\infty} \psi(e^{-x\alpha})e^{x\alpha} = \lim_{t\to+0} \frac{\psi(t)}{t} = \psi'(0) = \frac{1}{c_\alpha}. \tag{4.21}$$

Finally, splitting the integral in the definition of c_α and making the change variables $\tau = (1 - t)^\alpha$, we obtain

$$\begin{aligned} \ln\frac{1}{c_\alpha} &= \lim_{s\to+0} \int_s^1 \Big(\frac{\alpha}{1 - (1 - x)^\alpha} - \frac{1}{x}\Big)dx = \lim_{s\to+0} \Big(\int_0^{(1-s)^\alpha} \frac{\tau^{\frac{1-\alpha}{\alpha}} d\tau}{1 - \tau} + \ln s\Big) \\ &= \lim_{s\to+0} \Big(-\int_0^{(1-s)^\alpha} \frac{1 - \tau^{\frac{1-\alpha}{\alpha}}}{1 - \tau} d\tau - \ln\frac{1 - (1 - s)^\alpha}{s}\Big) \\ &= -\ln\alpha - \int_0^1 \frac{1 - \tau^{\frac{1-\alpha}{\alpha}}}{1 - \tau} d\tau = \ln\gamma_\alpha. \end{aligned}$$

Thus, combining the latter identity with (4.20), and (4.21) we obtain $c = 0$, which yields (4.19) and the mentioned analytic continuation property follows.

Now we prove (ii) and (iii). We note that in view of (4.14)

$$(\alpha t\phi_\alpha')' = \alpha(1 - \phi_\alpha)^{\alpha-1}\phi_\alpha' = \alpha\phi_\alpha' \frac{1 - \alpha t\phi_\alpha'}{1 - \phi_\alpha},$$

which implies

$$t\phi''_\alpha = \phi_\alpha(t\phi''_\alpha + \phi'_\alpha) - \alpha t\phi'^2_\alpha. \tag{4.22}$$

Setting

$$\phi_\alpha(t) = \sum_{k=1}^{\infty} a_k t^k \tag{4.23}$$

for the Taylor series of ϕ_α around $t = 0$ (we recall that $\phi_\alpha(0) = 0$) we obtain after comparison of the corresponding coefficients for all $k \geq 2$,

$$a_k = \frac{1}{k(k-1)} \sum_{\nu=1}^{k-1} a_\nu a_{k-\nu}[(1+\alpha)\nu - \alpha k]\nu = \sigma_k a_1^k.$$

Here $a_1 = \phi'_\alpha(0) = 1/c_\alpha = \gamma_\alpha$ and σ_k are defined as in (4.11). This yields the desired Taylor expansion. Moreover, we show that for $\alpha \in (0,1)$ the coefficients $\sigma_k > 0$ for all $k \geq 1$. Indeed,

$$\sigma_k = \frac{1}{2k(k-1)} \sum_{\nu=1}^{k-1} A_{\nu,k-\nu}\sigma_\nu\sigma_{k-\nu},$$

where

$$\begin{aligned} A_{\nu,k-\nu} &= [(1+\alpha)\nu - \alpha k]\nu + [(1+\alpha)(k-\nu) - \alpha k](k-\nu) \\ &= (1+\alpha)\left(\nu - \frac{k}{2}\right)^2 + \frac{1-\alpha}{2}k^2 > 0, \end{aligned}$$

unless $k = 2\nu$ when we also have

$$A_{\nu,\nu} = 2[(1+\alpha)\nu - 2\nu\alpha]\nu = 2(1-\alpha)\nu^2 > 0.$$

Since $\sigma_1 = 1$ and for $k \geq 1$ the coefficients $A_{\nu,k-\nu}$ before $\sigma_\nu\sigma_{k-\nu}$ are positive, the positiveness of σ_k follows now by induction.

Thus, $\phi_\alpha(t)$ has the Taylor expansion with positive coefficients. By standard facts of the power series theory we conclude that the radius R of convergence of (4.23) is at least $R = 1$ since $\phi_\alpha(t)$ is analytic along $t \in (-\epsilon, 1)$.

It remains only to prove (iv). We have $\phi'(0) > 0$ which yields $\phi_\alpha(t) < 0$ for sufficiently small $t < 0$. Then a standard analysis of (4.14) shows that these property holds for *all* negative t's where $\phi_\alpha(t)$ is defined. In view of (4.14), this proves the strictly increasing character of $\phi_\alpha(t)$.

In order to prove convexity, we note that (4.22) implies

$$\phi''_\alpha(t) = \phi'_\alpha(t)\frac{\phi_\alpha(t) - \alpha t\phi'_\alpha(t)}{t(1-\phi_\alpha(t))}, \quad \phi''(0) = (1-\alpha)\phi'^2_\alpha(0) > 0.$$

Clearly, it suffices to prove that $\phi''(t) \neq 0$. Assuming the contradictory, we have $\phi''(t) = 0$ an some point $t \neq 0$, and it follows that

$$\phi_\alpha(t) - \alpha t\phi'_\alpha(t) = 0,$$

which yields $(1 - \phi_\alpha(t))^\alpha = 1 - \phi_\alpha(t)$. The contradiction is obtained.

Finally, since $\phi_\alpha(t)$ is convex and analytic in its region of definition, we conclude that it can be infinitely extended into the left side of $\mathbb{R}$. □

5. Final remark

Here we discuss in short an appearance of the profile function $\mathcal{M}_1(w)$ as interpretation of the exponential transform. We recall that the original result of A.A. Markov on the L-problem asserts that a sequence of reals $\{s_j\}_{j=0}^{\infty}$ is represented as the moments

$$s_k = \int x^k \rho(x)dx$$

of certain function $0 \le \rho(x) \le 1$, if and only if there is a positive measure $d\mu$ such that the following identity holds

$$1 - \exp(-\sum_{k=0}^{\infty} \frac{s_k}{z^{k+1}}) = \sum_{k=0}^{\infty} \frac{a_k}{z^{k+1}},$$

where

$$a_k = \int x^k d\mu(x).$$

For the detailed discussion of this theory see [2, p. 72]. The latter moment sequence, $\{a_j\}_{j=0}^{\infty}$, can be characterized as a standard *positive* sequence in the sense that the Hankel forms

$$(a_{i+j})_{i,j=0}^{m} \ge 0$$

are positive semi-definite for all $m \ge 0$. For simplicity reasons, we refer to (s_k) as an L-sequence.

Given a sequence $(a_k)_{k=0}^{\infty}$ we set

$$\widehat{a}(z) := \sum_{k=0}^{\infty} \frac{a_k}{z^{k+1}}$$

for the corresponding z-transform. Our first observation is as follows.

Proposition 5.1. *Let $c \in \mathbb{R}$, and $\{a_j\}_{j=0}^{\infty}$ and $\{b_j\}_{j=0}^{\infty}$ be two sequences such that their generating functions satisfy*

$$\frac{1}{\widehat{b}(z)} - \frac{1}{\widehat{a}(z)} = c. \tag{5.1}$$

Then $\{a_j\}_{j=0}^{\infty}$ is a positive sequence if and only $\{b_j\}_{j=0}^{\infty}$ is. Moreover, we have

$$\det(a_{i+j})_{i,j=0}^{m} = \det(b_{i+j})_{i,j=0}^{m}. \tag{5.2}$$

Proof. We prove only (5.2) since it immediately implies the desired positivity property. Let for definiteness, $\widehat{a}(z)$ satisfies the positivity condition, i.e., the corresponding sequence (a_k) is positive semi-definite.

Then the famous result of Stieltjes [12, Ch. XI] asserts that given a function $\widehat{a}(z)$ with power series as above, the following continued J-fraction (actually, Jacobi's type) decomposition holds

$$\widehat{a}(z) = \cfrac{\alpha_0}{\beta_1 + z - \cfrac{\alpha_1}{\beta_2 + z - \cfrac{\alpha_2}{\beta_3 + z - \dots}}}. \tag{5.3}$$

Moreover, in this case we have for the determinants

$$\det(a_{i+j})_{i,j=0}^{m} = \alpha_0^{m+1}\alpha_1^{m}\alpha_2^{m-1}\cdots\alpha_{m-1}^{2}\alpha_m. \tag{5.4}$$

Now, it follows from (5.1) and (5.3) that

$$\widehat{b}(z) = \cfrac{1}{c + \cfrac{1}{\widehat{a}(z)}} = \cfrac{\alpha_0}{c\alpha_0 + \beta_1 + z - \cfrac{\alpha_1}{\beta_2 + z - \cfrac{\alpha_2}{\beta_3 + z - \dots}}}.$$

The latter continuous fraction is the Stieltjes' J-fraction for $\widehat{b}(z)$ and hence we have for its determinants the same expressions as those in (5.4), and (5.2) follows. □

Corollary 5.2. *The sequence $\{s_j\}_{j=0}^{\infty}$ is an L-sequence if and only if*

$$\mathcal{M}_1\left(\frac{1}{2}\widehat{s}(z)\right) = \widehat{b}(z) \tag{5.5}$$

for some positive sequence $\{b_j\}_{j=0}^{\infty}$.

Proof. Indeed, we have

$$\mathcal{M}_1(w) = \tanh w = \frac{e^{2w}-1}{e^{2w}+1},$$

therefore,

$$\widehat{b}(z) \equiv \mathcal{M}_1\left(\frac{1}{2}\widehat{s}(z)\right) = \frac{1-v(z)}{1+v(z)},$$

where $v(z) = \exp(-\widehat{s}(z))$ is the standard exponential transform of $\widehat{s}(z)$. Since $1 - v(z)$ is the generating function of some positive sequence (a_k), we have

$$\widehat{b}(z) = \frac{\widehat{a}(z)}{2-\widehat{a}(z)},$$

or

$$\frac{1}{\widehat{b}(z)} = \frac{2}{\widehat{a}(z)} - 1,$$

and the required property follows from positivity of $\widehat{a}(z)/2$. □

Remark 5.3. The previous observation makes it possible to consider an analogue of the (n-dimensional) transform by letting

$$\mathbb{E}_\rho^n(x) := 1 - \mathcal{M}_n(V_\rho(x)).$$

In particular, $\mathbb{E}_\rho^2(x) = E_\rho(x)$, while for $n = 1$ we have

$$\mathbb{E}_\rho^1(x) = \frac{2E_\rho(x)}{1+E_\rho(x)}.$$

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