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Ullemar's Formula for the Jacobian of the Complex Moment Mapping

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The complex moment sequence $\mu(P)$ is assigned to a univalent polynomial $P(z)$ by the Cauchy transform of the domain $P(U)$, where U is the unit disk. We establish the representation of the Jacobian $\det d\mu(P)$ in terms of roots of the derivative $P'(z)$. Combining this result with the special decomposition for the Hurwitz determinants, we prove a formula for $\det d\mu(P)$, which was previously conjectured by Ullemar. As a consequence, we show that the boundary of the class of all locally univalent polynomials in $\overline{U}$ is contained in the union of three irreducible algebraic surfaces.

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1. INTRODUCTION

Let $f(z), f(0)=0$, be an analytic function defined in a neighborhood of the unit disk U and $k \geq 0$ be a nonnegative integer. Then the *complex moments* of $f(z)$ are defined by

$$M_k(f) = \frac{i}{2\pi} \iint_U f^k(z) |f'(z)|^2 \, dz \wedge d\bar{z}.$$

This notion appears in several problems of complex analysis and its applications. In particular, if $f(z)$ is a univalent function in U, then the latter sequence constitutes an infinite family of invariants of the Hele–Shaw problem [12]. On the other hand, the sequence $(M_k(f))_{k\geq 0}$ defines the germ at infinity of the Cauchy transform of the domain $D = f(U)$:

$$\hat{\chi}_D = \frac{i}{2\pi} \iint_D \frac{d\zeta \wedge d\bar{\zeta}}{z-\zeta} = \sum_{k\geq 0} \frac{M_k(f)}{z^{k+1}}.$$

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Since the above definition may be regarded as a two-dimensional extension of the Stieltjes moments on the real line [2], it makes natural the corresponding inverse problem of defining $f(z)$ by its moment sequence. It follows from the result of Sakai [13] that without any additional restrictions, $f(z)$ (or the domain D) cannot be uniquely determined by its moments. Some recent results concerning the reconstruction of a domain by its complex moments can be found in [7,8,10].

Throughout this article we suppose that $f(z)$ is a polynomial

$$P(z) = a_1 z + \cdots + a_n z^n, \quad a_1 > 0, \tag{1}$$

of degree $n \geq 2$. Then $P(U)$ is an example of quadrature domain (see [1] and [4, p. 11]). It follows from a formula of Richardson (see (9)) that in this case the corresponding sequence $(M_k(P))_{k\geq 0}$ is finite and

$$M_k(P) = 0, \qquad k \geq n = \deg P. \tag{2}$$

Moreover,

$$M_0(P) = \sum_{j=1}^{n} j|a_j|^2 > 0, \qquad M_{n-1}(P) = a_1^n \overline{a}_n \neq 0.$$

Then it follows from (2) and Richardson's formula (9) that the complex moment sequence induces the *moment mapping* as a polynomial mapping

$$\mu_{\mathbb{C}}(P) = (M_0(P), \ldots, M_{n-1}(P)) : \mathbb{R}^+ \times \mathbb{C}^{n-1} \to \mathbb{R}^+ \times \mathbb{C}^{n-1}. \tag{3}$$

Similarly, in the case of the *real* polynomials $P(z)$, i.e. $a_k \in \mathbb{R}$, (3) induces a real polynomial mapping

$$\mu(P) = (M_0(P), \ldots, M_{n-1}(P)) : \mathbb{R}^+ \times \mathbb{R}^{n-1} \to \mathbb{R}^+ \times \mathbb{R}^{n-1}. \tag{4}$$

Thus the above mentioned inverse problem can be reformulated as an injectivity problem for the preceding polynomial mappings.

Let $\mathcal{P}_n(\overline{U})$ denote the class of all polynomials (1) univalent in a neighborhood of the closed unit disk, $a_k \in \mathbb{R}$, $1 \leq k \leq n$. By $\mathcal{P}_{n,\mathrm{loc}}(\overline{U})$ we denote the class of the locally univalent polynomials; equivalently, $P(z) \in \mathcal{P}_{n,\mathrm{loc}}(\overline{U})$ iff $P'(z) \neq 0$ in $\overline{U}$. It is clear, that $\mathcal{P}_n(\overline{U})$ is a proper subclass of $\mathcal{P}_{n,\mathrm{loc}}(\overline{U})$ for $n \geq 3$.

The main difficulty in the study of the injectivity problem of μ and $\mu_{\mathbb{C}}$ is a highly involved structure of the class of univalent polynomials $\mathcal{P}_n(\overline{U})$. Only some low degree ($n \leq 3$) results are known (see [9,3,15]).

It was proven by Ullemar in [16] that μ is *globally* injective on $\mathcal{P}_3(\overline{U})$ and the injectivity property fails on $\mathcal{P}_{3,\mathrm{loc}}(\overline{U})$. The first general result for the *locally* univalent polynomials (actually, even with complex coefficients) is due to Gustafsson [6] and states that μ is locally injective on $\mathcal{P}_{n,\mathrm{loc}}(\overline{U})$. The question whether μ is *globally* injective on $\mathcal{P}_n(\overline{U})$ for $n \geq 4$ is still open.

In her paper, Ullemar conjectured the following formula for the Jacobian of μ:

$$J(P) \equiv \det d\mu(P) = 2^{-n(n-3)/2} {a_1}^{n(n-1)/2} P'(1)P'(-1)\Delta_n(\widetilde{P}'(z)), \tag{5}$$

which will be in focus in the present article. Here $\Delta_n(\widetilde{P}'(z))$ denotes the main Hurwitz determinant for the Möbius transformation of the derivative $P'(\zeta)$ (see exact definitions in Section 4).

A useful feature of (5) is that it immediately implies the local injectivity property. Indeed, by the well-known Hurwitz theorem the inner determinant in (5) is positive when $P'(z)$ has no roots in a right half-plane.

Our first result gives the following alternative formula for evaluation of $J(P)$ via the inner characteristics of P.

THEOREM 1 (Derivative Roots Formula) *Let $P(z) = a_1 z + \cdots + a_n z^n$, $a_k \in \mathbb{R}$ and $\zeta_1, \ldots, \zeta_{n-1}$ are all zeroes of the derivative $P'(z)$. Then*

$$\begin{aligned} J(P) &= 2a_1^{n(n-1)/2}(na_n)^n \cdot \prod_{i \leq j}(\zeta_i\zeta_j - 1) \\ &= 2a_1^{n(n-1)/2}(na_n)^{n-2}\, P'(1)\, P'(-1) \prod_{i<j}(\zeta_i\zeta_j - 1). \end{aligned} \tag{6}$$

Actually, the right hand side of (6), as a symmetric function of the roots, can be represented as a homogeneous form

$$J(P) = 2a_1^{n(n-1)/2} V_{n-1}(b_1, \ldots, b_n) \sum_{j=1}^{n} b_j \sum_{k=1}^{n} (-1)^k b_k,$$

where $b_k = ka_k$ are the coefficients of $P'(z)$ and V_{n-1} is a homogeneous irreducible polynomial of degree $(n-1)$ (see Section 6 for precise definitions).

THEOREM 2 (Resultant Formula) *Let $A^*(z) = z^p A(1/z)$ be the reciprocal polynomial to $A(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_p z^p$. Then*

$$J(P)^2 = 4(-1)^{n-1} a_1^{n(n-1)} \mathcal{R}(P', P'^*) \cdot P'(-1)P'(1),$$

where $\mathcal{R}(A, B)$ denotes the resultant of the corresponding polynomials.

Now, the Ullemar formula (5) can be obtained as a consequence of Theorem 1 and certain auxiliary properties of the Hurwitz determinants which we get in Section 4.

As another application we give an alternative proof of the above mentioned result of Gustafsson.

COROLLARY 1 *The mapping $\mu(P)$ is locally injective on the set $\mathcal{P}_{n,\mathrm{loc}}(\overline{U})$, $n \geq 1$.*

Proof Indeed, given any polynomial $P(z) \in \mathcal{P}_{n,\mathrm{loc}}(\overline{U})$ with real coefficients we have $a_n \neq 0$ and $a_1 = P'(0) \neq 0$. Moreover, $|P'(\zeta)| \neq 0$ in $\overline{U}$ and it follows that all zeroes

ζ_k of the first derivative satisfy $|\zeta_k| > 1$, $k = 1, \ldots, n-1$. Thus (6) implies that $J(P) \neq 0$. ■

It turns out that $J(P)$ can be associated with the structural properties of the class $\mathcal{P}_{n,\mathrm{loc}}(\overline{U})$ as follows. Let us identify a polynomial $P(z) = \sum_{j=1}^{n} a_j z^j$ with the point $(a_1, \ldots, a_n) \in \mathbb{R}^n$ and put

$$\mathcal{P}_{\mathrm{loc}}^{n} = \cup_{1 \leq j \leq n} \mathcal{P}_{j,\mathrm{loc}}(\overline{U}).$$

THEOREM 3 *Let $n \geq 3$, then the boundary of $\mathcal{P}_{\mathrm{loc}}^{n}$ is contained in the union of the following three irreducible algebraic varieties: the hyperplanes*

$$\begin{aligned} \Pi^{+}: &\quad P'(1) = a_1 + 2a_2 + \cdots + na_n = 0, \\ \Pi^{-}: &\quad P'(-1) = a_1 - 2a_2 + \cdots + (-1)^{n-1} na_n = 0, \end{aligned} \tag{7}$$

and an algebraic surface of $(n-1)$th order given by

$$\mathcal{A}: \quad V_{n-1}(a_1, 2a_2, \ldots, na_n) = 0. \tag{8}$$

It follows from the preceding results that $\mathcal{P}_{\mathrm{loc}}^{n}$ is exactly an open component of the set $\{P\colon J(P) \neq 0\}$.

The similar result for the univalent classes $\mathcal{P}_n(\overline{U})$ is due to Quine [11]. But only upper estimates for the degree of the boundary $\partial\mathcal{P}_n(\overline{U})$ have been established there.

We notice that the previous formulae as well as the suitable modifications of basic facts below are still valid for polynomials with complex coefficients. This will be accomplished in a forthcoming paper.

2. PRELIMINARY RESULTS

Following to Richardson [12] one can write the following expressions for $M_k(P)$

$$M_k(P) = \sum i_1 a_{i_1} \cdots a_{i_{k+1}} \overline{a}_{i_1 + \cdots + i_{k+1}}, \tag{9}$$

where the sum is taken over all possible sets of indices $i_1, \ldots, i_k \geq 1$. It is assumed that $a_j = 0$ for $j \geq n+1$. These formulae are easy to use for straightforward manipulations with the complex moments and it follows that $\mu_{\mathbb{C}}$ as well as μ are polynomial mappings. Nevertheless, this representation is useless for the further study of analytic properties of μ.

We shall use in the sequel the following simple residue representation of the moment sequence for real polynomials

$$M_k(P) = \frac{1}{k+1} \operatorname*{res}_{\zeta=0} \left(P^{k+1}(\zeta) P'\left(\frac{1}{\zeta}\right) \frac{1}{\zeta^2} \right). \tag{10}$$

Indeed, it follows from Stokes' formula that

$$\frac{i}{2\pi}\iint_G w^k\,dw\wedge d\overline{w} = \frac{i}{2\pi(k+1)}\int_{\partial G} w^{k+1}\,d\overline{w}, \tag{11}$$

where G is an arbitrary 2-chain in the complex plane. Letting $G = P(U)$ and taking into account that $\bar{\zeta} = \zeta^{-1}$ on ∂U, and the fact that $\overline{P'(z)} = P'(\overline{z})$ for polynomials with real coefficients, we obtain from (11)

$$M_k(P) = \frac{i}{2\pi(k+1)}\int_{\partial U} P^{k+1}(\zeta)\overline{P'(\zeta)}\,d\bar{\zeta} = \frac{1}{2\pi i(k+1)}\int_{\partial U} P^{k+1}(\zeta)P'\left(\frac{1}{\zeta}\right)\frac{d\zeta}{\zeta^2}.$$

This proves (10).

Moreover, since $P(0) = 0$, it follows that $P(\zeta) = zP_1(z)$, where P_1 is a polynomial. Thus, the expression

$$P'\left(\frac{1}{\zeta}\right)P^{k+1}(\zeta)\frac{1}{\zeta^2} = \zeta^{k-n}(a_1\zeta^{n-1} + \cdots + na_n)(a_1 + \cdots + a_n\zeta^{n-1})^{k+1}$$

is also a polynomial for all $k \geq n$ and it follows from (10) that

$$M_k(P) = \frac{1}{k+1}\operatorname*{res}_{\zeta=0}\zeta^{k+1}P_1^{k+1}(\zeta)P'\left(\frac{1}{\zeta}\right) = 0$$

which proves (2). Therefore, the mapping μ in (4) is well defined.

Given two meromorphic functions H_1 and H_2 we write

$$H_1(z) \equiv H_2(z) \qquad \operatorname{mod}[m_1; m_2]$$

if the Laurent series of $H_2 - H_1$ does not contain z^m with $m_1 \leq m \leq m_2$.

LEMMA 1 *For any* k, $0 \leq k \leq n-1$,

$$P'(z)\left(P^k(z) + P^k\left(\frac{1}{z}\right)\right) \equiv \sum_{\nu=1}^{n}\frac{\partial M_k(P)}{\partial a_\nu}\cdot z^{\nu-1} \quad \operatorname{mod}[0; n-1]. \tag{12}$$

Proof Let $\lambda_m(f(z)) = \operatorname{res}_{z=0}(f(z)z^{-1-m})$; then it follows from the relations

$$\frac{\partial P(1/z)}{\partial a_\nu} = \frac{1}{z^\nu}, \qquad \frac{\partial P'(z)}{\partial a_\nu} = \nu z^{\nu-1},$$

and (10) that

$$\frac{\partial M_k(P)}{\partial a_\nu} = \lambda_0\big(P^k(1/z)P'(z)z^{1-\nu}\big) + \frac{\nu}{k+1}\lambda_0\big(P^{k+1}(1/z)z^\nu\big). \tag{13}$$

On the other hand, integrating by parts yields

$$\lambda_0\big(z^\nu P^{k+1}(1/z)\big) = \frac{1}{2\pi i}\int_{\partial U} P^{k+1}(1/z)z^{\nu-1}\,dz = \frac{1}{2\pi i\nu}\int_{\partial U} d\big(z^\nu P^{k+1}(1/z)\big)$$
$$+\frac{k+1}{2\pi i\nu}\int_{\partial U} P^k(1/z)P'(1/z)z^{\nu-2}\,dz = \frac{k+1}{\nu}\lambda_0\big(P^k(1/z)P'(1/z)z^{\nu-1}\big),$$

and taking into account that $\lambda_0(f(1/z)) = \lambda_0(f(z))$ we arrive at

$$\lambda_0\big(P^{k+1}\big(z^{-1}\big)z^\nu\big) = \frac{k+1}{\nu}\lambda_0\big(P^k(z)P'(z)z^{1-\nu}\big). \tag{14}$$

Combining (14) and (13), we get

$$\frac{\partial M_k(P)}{\partial a_\nu} = \lambda_0\Big[P'(z)z^{1-\nu}\Big(P^k(z)+P^k\big(z^{-1}\big)\Big)\Big]$$
$$= \lambda_{\nu-1}\Big[P'(z)\Big(P^k(z)+P^k\big(z^{-1}\big)\Big)\Big]$$

and the required formula (12) follows. ∎

We notice that for any index $k \in \{0,\dots,n-1\}$ the following expansion

$$P^k(z)+P^k\big(z^{-1}\big) = \sum_{m=-nk}^{nk} h_m^{(k)} z^m, \tag{15}$$

yields the symmetry property: $h_m^{(k)} = h_{-m}^{(k)}$.

To study (12) it is convenient to consider a slightly more general case. Namely, given an arbitrary vector $x = (x_0, x_1, \dots, x_{n-1})$, we define the following Toeplitz matrix

$$\mathcal{T}(x) = \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_1 & x_0 & \cdots & x_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & \cdots & \cdots & x_0 \end{pmatrix}.$$

Then we can introduce the *dual* matrix $\mathcal{B}(y)$, $y = (y_0, y_1, \dots, y_{n-1})$, by

$$\mathcal{T}(x)\cdot y^\top = \mathcal{B}(y)\cdot x^\top, \quad \forall x \in \mathbb{R}^n. \tag{16}$$

Unlike $\mathcal{T}(x)$, the matrix $\mathcal{B}(y)$ is not symmetric and has a more complicated structure. We shall study $\mathcal{B}(y)$ in more detail in the next section.

Let $H_k(z)$ be rational functions having Laurent series of the form

$$H_k(z) = \sum_{m=-N}^{N} h_{|m|}^{(k)}\, z^m,$$

and let

$$B(z) = b_0 + b_1 z + \cdots + b_{n-1} z^{n-1}$$

be a polynomial such that $b_{n-1} \neq 0$.

Then we can define polynomials

$$\Phi_k(z) = \sum_{\nu=0}^{n-1} \varphi_\nu^{(k)} z^\nu, \quad 0 \le k \le n-1$$

such that

$$B(z) \cdot H_k(z) \equiv \Phi_k(z) \quad \mathrm{mod}[0; n-1]. \tag{17}$$

Further, we consider the vectors $h^{(k)} = (h_0^{(k)}, \ldots, h_{n-1}^{(k)})$ and $b = (b_0, \ldots, b_{n-1})$. It follows then from (17) that the following matrix identity holds

$$(\varphi_0^{(k)}, \ldots, \varphi_{n-1}^{(k)})^\top \equiv \varphi^{(k)^\top} = \mathcal{T}(h^{(k)}) \cdot b^\top,$$

which by virtue of (16) implies $\varphi^{(k)^\top} = \mathcal{B}(b) \cdot h^{(k)^\top}$, $0 \le k \le n-1$. Therefore, denoting by Φ and H the matrices formed by combination of the columns $\varphi^{(k)^\top}$ and $h^{(k)^\top}$ respectively, we get $\Phi = \mathcal{B}(b)H$ and

$$\det \Phi = \det \mathcal{B}(b) \cdot \det H. \tag{18}$$

To apply the preceding arguments to our case we let $B(z) = P'(z)$ and $H_k(z) = P^k(z) + P^k(1/z)$. Hence, we obtain from (12)

$$\varphi_\nu^{(k)} = \frac{\partial M_k(P)}{\partial a_\nu}, \qquad d\mu(P) = \Phi. \tag{19}$$

Thus, the problem of evaluating the Jacobian $J(P)$ can be reduced, by virtue of (18), to the corresponding problem for the determinants of $\mathcal{B}(b)$ and H (here $b_{j-1} = ja_j$ corresponds to the coefficients of $P'(z)$).

The latter determinant can be found as follows. First note that $\|h_i^{(k)}\|$ is a lower triangular matrix in our case. Indeed, we have $P(z) = zP_1(z)$, where $P_1(z)$ is a polynomial, and it follows that

$$P^k(z) + P^k\left(z^{-1}\right) = z^k P_1^k(z) + \frac{1}{z^k} P_1^k\left(z^{-1}\right) = \sum_{m=k}^{kn} (z^m + z^{-m}) h_m^{(k)}.$$

This representation easily implies that $h_m^{(k)} = 0$, where $0 \le m \le k-1$. Moreover, we have for the diagonal elements $h_0^{(0)} = 2$ and $h_k^{(k)} = a_1^k$. This yields

$$\det H = \det \|h_i^{(k)}\| = 2 \cdot a_1 \cdot a_1^2 \cdot \cdots \cdot a_1^{n-1} = 2a_1^{n(n-1)/2}. \tag{20}$$

3. TOEPLITZ DETERMINANTS

The explicit expression of $\det \mathcal{B}(y)$ in terms of the coefficients $y_0, \ldots, y_m$ is messy and useless for the further analysis. However, it turns out that this determinant can be easily written in terms of certain intrinsic characteristics of y. Namely, let us associate with any vector $y \in \mathbb{R}^m$ the polynomial

$$B_y(z) = y_0 + y_1 z + \cdots + y_m z^m, \quad y_m \neq 0.$$

THEOREM 4. *Let $\zeta_1, \ldots, \zeta_m$ be the roots of $B_y(\zeta)$ counted according to their multiplicities. Then*

$$\det \mathcal{B}(y) = y_m^{m+1} \prod_{i \geq j} (\zeta_i \zeta_j - 1), \tag{21}$$

Proof First note that the left-hand side of (21) is an algebraic function of $y_0, \ldots, y_m$ and, hence, it is sufficient to prove (21) for any $\zeta = (\zeta_1, \ldots, \zeta_m)$ outside a proper algebraic submanifold of $\mathbb{C}^m$. Namely, we will suppose that $\zeta_i \neq \zeta_j$ for $i \neq j$ and $\zeta_i \zeta_j \neq 1$ for all i, j.

Given a nonnegative integer k and $\zeta \in \mathbb{C}$ we define the following vector

$$\{\zeta\}_k = (0, \ldots, 0, 1, \zeta, \zeta^2, \ldots, \zeta^{m-k})^\top \in \mathbb{C}^{m+1}, \qquad \{\zeta\} \equiv \{\zeta\}_0.$$

Then letting $x = \{\zeta\}^\top$ in (16) we get

$$\mathcal{T}(\{\zeta\}^\top) \cdot y^\top = B_y(\zeta) \cdot \{\zeta^{-1}\} + \sum_{i=0}^{m-1} y_i \big(\{\zeta\}_i - \{\zeta^{-1}\}_i\big)$$

and changing ζ by ζ^{-1} in the preceding formula we arrive after summation at

$$\mathcal{T}(\{\zeta\}^\top + \{\zeta^{-1}\}^\top) \cdot y^\top = B_y(\zeta) \cdot \{\zeta^{-1}\} + B_y(\zeta^{-1}) \cdot \{\zeta\}. \tag{22}$$

Let $\zeta = \zeta_i$ be a root of $B_y(\zeta)$; then it follows from (22) that

$$\mathcal{T}(\{\zeta_i\}^\top + \{\zeta_i^{-1}\}^\top) \cdot y^\top = B_y(\zeta_i^{-1}) \cdot \{\zeta_i\} \tag{23}$$

and

$$\mathcal{T}(e) \cdot y^\top = B_y(1) \cdot e^\top, \tag{24}$$

where $e = (2, \ldots, 2) \in \mathbb{C}^{m+1}$. Applying (16) to the left-hand sides of (23) and (24) we obtain

$$\mathcal{B}(y)(\{\zeta_i\} + \{\zeta_i^{-1}\}) = B_y(\zeta_i^{-1}) \cdot \{\zeta_i\}, \quad \forall i = 1, \ldots, m,$$

and

$$\mathcal{B}(y) \cdot e^{\top} = B_y(1) \cdot e^{\top}.$$

Combining the preceding expressions into the matrix form we arrive at the following relation for determinants

$$\begin{aligned}
&\det \mathcal{B}(y) \det \mathcal{W}(1, \zeta_1, \ldots, \zeta_m) \\
&= 2B_y(1) \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & \zeta_1 & \ldots & \zeta_1^m \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_m & \ldots & \zeta_m^m \end{pmatrix} \prod_{j=1}^m B_y(\zeta_j^{-1}) \\
&= 2(-1)^m B_y(1) \prod_{k=1}^m B_y(\zeta_k^{-1}) \cdot \prod_{i<j} (\zeta_j - \zeta_i) \cdot \prod_{i=1}^m (1 - \zeta_i), \qquad (25)
\end{aligned}$$

where $\mathcal{W}(\alpha_0, \alpha_1, \ldots, \alpha_m)$ denotes the matrix with the entries

$$\mathcal{W}_{ij} = \| \alpha_j^i + \alpha_j^{-i} \|_{i,j=0}^m.$$

The determinant of $\mathcal{W}(\alpha_0, \ldots, \alpha_m)$ can be found by the same method as the Vandermonde determinant (see also [17, Part 4]):

$$\det \mathcal{W}(\alpha_0, \ldots, \alpha_m) = \frac{2}{(\alpha_0 \ldots \alpha_m)^m} \prod_{i<j} (\alpha_j - \alpha_i) \prod_{i<j} (\alpha_i \alpha_j - 1)$$

and it follows that

$$\det \mathcal{W}(1, \zeta_1, \ldots, \zeta_m) = \frac{2}{(\zeta_1 \cdots \zeta_m)^m} \prod_{i<j} (\zeta_j - \zeta_i) \prod_{i<j} (\zeta_i \zeta_j - 1) \prod_{j=1}^m (1 - \zeta_j)^2. \qquad (26)$$

On the other hand,

$$\prod_{k=1}^m B_y(\zeta_k^{-1}) = (-1)^m \frac{y_m^m}{(\zeta_1 \cdots \zeta_m)^m} \prod_{j=1}^m \prod_{i=1}^m (\zeta_i \zeta_j - 1).$$

Thus, applying the previous identities to (25) we obtain

$$\det \mathcal{B}(y) = \frac{B_y(1) y_m^m}{\prod_{i=1}^m (1 - \zeta_i)} \prod_{j \geq i} (\zeta_i \zeta_j - 1)$$

which implies by virtue of

$$B_y(1) = y_m \prod_{i=1}^{m}(1-\zeta_i)$$

the required identity. ■

Proof of Theorem 1 It follows from (19) that $d\mu(P) = \Phi$. Then applying (20) and Theorem 4 to (18) we obtain

$$J(P) \equiv \det\left[\frac{\partial M_k(P)}{\partial a_i}\right] = 2a_1^{n(n-1)/2} \cdot b_{n-1}^n \prod_{i\le j}(\zeta_i\zeta_j - 1),$$

where $b_{n-1} = na_n$ is the leading coefficient of $B(z) \equiv P'(z)$ and the theorem follows. ■

4. HURWITZ DETERMINANTS AND ULLEMAR'S FORMULA

Let us consider an arbitrary polynomial $R(z) = r_0 + r_1 z + \cdots + r_m z^m$ of degree $m \geq 1$. Let us extend the sequence of the coefficients r_k such that $r_k = 0$ for all $k > m$ and $k < 0$. Then the $m \times m$-matrix

$$\mathcal{G}(R) \equiv \begin{pmatrix} r_{m-1} & r_{m-3} & \dots & r_{1-m} \\ r_m & r_{m-2} & \dots & r_{2-m} \\ \vdots & \vdots & \ddots & \vdots \\ r_{2m-2} & r_{2m-4} & \dots & r_0 \end{pmatrix}$$

is called the Hurwitz matrix of the polynomial $R(z)$ [5]. More specifically, the entries of the matrix have the form

$$\mathcal{G}_{ij}(R) = r_{m+i-2j}. \tag{27}$$

The main diagonal minor $\Delta(R)$ of the $(m-1)$th order of $\mathcal{G}(R)$ is said to be the *Hurwitz determinant* of R. It immediately follows from the above definition that

$$\det \mathcal{G}(R) = r_0\, \Delta(R). \tag{28}$$

THEOREM 5 *The Hurwitz determinant of* $R(z)$, $\deg R = m$, *has the following representation*

$$\Delta(R) = (-1)^{(m^2-m)/2} r_m^{m-1} \prod_{1\le i<j\le m} (z_i + z_j), \tag{29}$$

where z_i *are the roots of* $R(z)$ *counted according to their multiplicity.*

Before we give the proof of the theorem let us formulate some of its corollaries. Let us consider the *Möbius transformation* of the polynomial $R(z)$ given by

$$\widetilde{R}(z) = (z+1)^m R\left(\frac{z-1}{z+1}\right) \equiv \widetilde{r}_0 + \widetilde{r}_1 z + \cdots + \widetilde{r}_m z^m.$$

Obviously, $\zeta_k = (1+z_k)/(1-z_k)$ are the roots of $\widetilde{R}(z)$ whenever $z_1, \ldots, z_m$ are the roots of $R(z)$. In particular, all the roots of $R(z)$ are contained in the unit disk if and only if the roots of $\widetilde{R}(z)$ lie in the right half-plane. Moreover,

$$\prod_{1 \le i < j \le m} (\zeta_i + \zeta_j) = 2^{m(m-1)/2} \prod_{1 \le i < j \le m} (1 - z_i z_j) \left(\prod_{i=1}^{m} (1 - z_i)\right)^{1-m}.$$

Then the following identities

$$\prod_{i=1}^{m} (1 - z_i) = \frac{R(1)}{r_m},$$

$$\widetilde{r}_m = \lim_{z \to \infty} z^{-m} \widetilde{R}(z) = R(1),$$

together with (29) yield

COROLLARY 2 *In the previous notations*

$$\Delta(\widetilde{R}) = 2^{(m^2-m)/2} r_m^{m-1} \prod_{1 \le i < j \le m} (z_i z_j - 1), \tag{30}$$

where r_m is the leading coefficient of R and $\{z_i\}_{1 \le i \le m}$ are the roots of R.

Now, Ullemar's conjectured formula (5) is a simple consequence of (30) and Theorem 1.

COROLLARY 3 (Ullemar Formula) *The Jacobian of the complex moment mapping μ has the following representation*

$$J(P) \equiv \det\, d\mu(P) = 2^{-n(n-3)/2} {a_1}^{n(n-1)/2} P'(1) P'(-1) \Delta_n(\widetilde{P}'(z)),$$

where $n = \deg P$.

Proof of Theorem 5 Similarly to the proof of Theorem 1 we can assume that $R(z)$ has no multiple roots. Then we have from (27) for any $\zeta \in \mathbb{C}$ and any index i, $1 \le i \le m$, that

$$\sum_{j=1}^{m} \mathcal{G}_{ij}(R) \zeta^{2m-2j} = \sum_{j=1}^{m} r_{m+i-2j} \zeta^{2m-2j} = \zeta^{m-i} \sum\nolimits_k^{\star} \zeta^k r_k,$$

where the kth index in the last sum has the form $k(i,j) = m + i - 2j$, $j = 1, \ldots, m$.

Now suppose that $i \in [1; m]$ and let $\bar{i} = m - i$. It is clear that k takes only the even (or only odd) values which range between $-\bar{i}$ and $2m - 2 - \bar{i}$ with changing j between 1 and m. Moreover, both k and $\bar{i}$ have the same parity and

$$-\bar{i} \le k(i,j) \le 2m - 2 - \bar{i}, \quad \bar{i} = 0, 1, \ldots, m - 1.$$

Hence, for each i from $[1, m]$ the indices $k(i,j)$ take all the values of $\bar{i}$ from interval $[0; m-1]$ when $j \in [0; m]$. By virtue of this property we conclude that

$$\sum_{j=1}^{m} \mathcal{G}_{ij}(R)\zeta^{2m-2j} = \zeta^{m-i} R^{[\bar{i}]}(\zeta), \tag{31}$$

where $R^{[p]}(\zeta)$ is the even part of $R(\zeta)$ for even p and the odd part of $R(\zeta)$ for odd p:

$$R^{[p]}(\zeta) = \frac{1}{2}\left(R(\zeta) + (-1)^p R(-\zeta)\right).$$

Let now $\zeta = z_k$ be the kth root of $R(\zeta)$. Taking into account that

$$R^{[p]}(z_k) = R(z_k) - R^{[p+1]}(z_k) = -R^{[p+1]}(z_k),$$

we obtain

$$R^{[p]}(z_k) = (-1)^p R^{\text{ev}}(z_k).$$

Here $R^{\text{ev}}(\zeta) = R^{[0]}(\zeta)$ is the even part of $R(\zeta)$ and we see from (31) that

$$\sum_{j=1}^{m} \mathcal{G}_{ij}(R) z_k^{2m-2j} = (-z_k)^{m-i} R^{\text{ev}}(z_k).$$

Combining the last identities for $k = 1, 2, \ldots, m$ into the matrices we obtain for their determinants

$$\det \mathcal{G}(R) \det \mathcal{V}(z_1^2, \ldots, z_m^2) = (-1)^{m(m-1)/2} \det \mathcal{V}(z_1, \ldots, z_m) \prod_{k=1}^{m} R^{\text{ev}}(z_k), \tag{32}$$

where $\mathcal{V}(a_1, \ldots, a_m) = \|a_j^{k-1}\|_{j,k=1}^{m}$ is the Vandermonde matrix.

On the other hand, we have for the even part

$$R^{\text{ev}}(z_k) = \frac{1}{2} R(-z_k) = (-1)^m \frac{r_m}{2} \prod_{i=1}^{m} (z_i + z_k),$$

and it follows from (32) that

$$\det \mathcal{G}(R) \cdot \prod_{1 \le i < j \le m} (z_j^2 - z_i^2) = \frac{(-1)^{m(m+1)/2} r_m^m}{2^m} \prod_{i,\, j=1}^{m} (z_i + z_j) \prod_{1 \le i < j \le m} (z_j - z_i).$$

Hence, applying (28) we find

$$\Delta(R) = \frac{(-1)^{(m^2+m)/2} r_m^m}{2^m r_0} \prod_{1 \le i \le j \le m} (z_i + z_j)$$

and rewriting the last product as

$$\prod_{1 \le i \le j \le m} (z_i + z_j) = \prod_{i=1} (2z_i) \prod_{1 \le i < j \le m} (z_i + z_j) = \frac{(-2)^m r_0}{r_m} \prod_{1 \le i < j \le m} (z_i + z_j),$$

we arrive at the required identity and the theorem is proved. ■

5. REPRESENTATIONS VIA THE RESULTANTS

Recall that given two polynomials

$$A(z) = A_n \prod_{k=1}^{n} (z - \alpha_k) = A_0 + A_1 z + \cdots + A_n z^n,$$

and

$$B(z) = B_n \prod_{k=1}^{n} (z - \beta_k) = B_0 + B_1 z + \cdots + B_n z^n,$$

the product

$$\mathcal{R}(A, B) = A_n^n B_n^n \prod_{i,j=1}^{n} (\alpha_i - \beta_j)$$

is called the *resultant* of A and B.

If $A(z)$ and $B(z)$ are the mutually reciprocal polynomials

$$B(z) = z^n A(1/z) \equiv A^*(z),$$

then $B_{n-j} = A_j, j = 0, \dots n$ and we have for their roots: $\beta_j = 1/\alpha_j$. Then the corresponding resultant can be rewritten in the matrix form

$$\mathcal{R}(A, A^*) = \det \begin{pmatrix} A_0 & A_1 & \dots & \dots & A_n & & & & \\ & A_0 & A_1 & \dots & \dots & A_n & & & \\ & & \dots & \dots & \dots & \dots & \dots & & \\ & & & A_0 & A_1 & \dots & \dots & A_n \\ A_n & A_{n-1} & \dots & \dots & A_0 & & & \\ & A_n & A_{n-1} & \dots & \dots & A_0 & & \\ & & \dots & \dots & \dots & \dots & \dots & \\ & & & A_n & A_{n-1} & \dots & \dots & A_0 \end{pmatrix}. \tag{33}$$

It is easy to see that the last matrix is of $2n$th order and with A_0 as its diagonal elements. On the other hand,

$$\mathcal{R}(A, A^*) = A_n^n A_0^n \prod_{i,j=1}^{n} \left(\alpha_i - \frac{1}{\alpha_j} \right) = \frac{A_n^n A_0^n}{(\alpha_1 \dots \alpha_n)^n} \prod_{i>j}^{n} \big(\alpha_i \alpha_j - 1\big)^2 \prod_{i=1}^{n} (\alpha_i^2 - 1)$$

and by Viète's theorem

$$\alpha_1 \cdots \alpha_n = (-1)^n \frac{A_0}{A_n}, \qquad A(1)A(-1) = A_n^2 \prod_{i=1}^{n} (\alpha_i^2 - 1),$$

we conclude that

$$\begin{aligned} \mathcal{R}(A, A^*) &= (-1)^n A(-1)A(1)A_n^{2n-2} \prod_{i>j}^{n} \big(\alpha_i \alpha_j - 1\big)^2 \\ &= \frac{(-1)^n A_n^{2n+2}}{A(1)A(-1)} \left[\prod_{i \geq j}^{n} \big(\alpha_i \alpha_j - 1\big) \right]^2. \end{aligned} \tag{34}$$

Thus, we have from (34)

$$W_n^2(A) = (-1)^n \mathcal{R}(A, A^*) A(-1)A(1) \tag{35}$$

where

$$W_n(A) = A_n^{n+1} \prod_{i \leq j} (\alpha_i \alpha_j - 1). \tag{36}$$

As an immediate consequence of its definition, $W_n(A) \equiv W_n(A_0, A_1, \ldots, A_n)$ is a homogeneous form of order $n+1 = \deg A + 1$. Moreover, it admits the following factorization

$$W_n(A) = A(-1)A(1)V_n(A), \qquad V_n(A) = A_n^{n-1}\prod_{i<j}(\alpha_i\alpha_j - 1), \tag{37}$$

where $V_n(A)$ is a homogeneous form of degree $(\deg A - 1)$.

On the other hand, it follows from

$$V_n(A) = A_0^{n-1}\prod_{i<j}\left(1 - \frac{1}{\alpha_i\alpha_j}\right), \tag{38}$$

that we have the recursion formula

$$V_n(A_0, A_1, \ldots, A_k, 0, \ldots, 0) = A_0^{n-k}V_k(A_0, A_1, \ldots, A_k).$$

Here are the explicit expressions for V_k:

$$\begin{aligned}
V_3(A) &= A_0^2 - A_0A_2 + A_1A_3 - A_3^2\\
V_4(A) &= A_4(-A_1^2 + A_3A_1 + A_4^2 - A_4A_2 - A_0A_4 + 2A_0A_2 - A_0^2)\\
&\quad + A_0(A_0^2 - A_0A_2 + A_1A_3 - A_3^2).
\end{aligned}$$

Proof of Theorem 2 Substituting the derivative

$$P'(z) = a_1 + 2a_2z + \cdots + na_nz^{n-1} \equiv b_1 + b_2z + \cdots + b_nz^{n-1}, \qquad b_k = ka_k,$$

as $A(z)$ to (34) and (35) we obtain

$$\left[b_n^n\prod_{i\geq j}^{n-1}(\zeta_i\zeta_j - 1)\right]^2 = (-1)^{n-1}\mathcal{R}(P', P'^*)P'(-1)P'(1). \tag{39}$$

Then comparing the last relations with the definition (36) we arrive at the following formula

$$W_{n-1}(P')^2 = (-1)^{n-1}\mathcal{R}(P', P'^*)P'(-1)P'(1).$$

Finally, combining the preceding identity with (6) we attain the required representation of $J(P)$

$$J^2(P) = 4b_1^{n^2-n}W_{n-1}^2(P') = 4b_1^{n^2-n}(-1)^{n-1}\mathcal{R}(P', P'^*)P'(-1)P'(1)$$

which with $a_1 = b_1$ completes the proof. ■

The following property of V_k will be used in the next section.

THEOREM 6 *$V_n(A) \equiv V_n(A_0, A_1, \ldots, A_n) \in \mathbb{C}[[A_0, A_1, \ldots, A_n]]$ is an irreducible polynomial.*

Proof A simple analysis of the denominator of the right-hand side of (38) shows that A_n cannot be a divisor of $V_n(A)$. On the other hand, we noticed that $V_n(A)$ can be represented as a symmetric polynomial function of the roots $(\alpha_k)_{1\le k\le n}$ of $A(z)=0$.

Let $H_1(A)$ and $H_2(A)$ be two nontrivial (i.e. different from the identical constants) divisors of $V_n(A)$. It is a consequence of the homogeneity of $V_n(A)$ that both of $H_k(A)$ are homogeneous too. Moreover, in our assumptions $h_k=\deg H_k\ge 1$.

By Viéte's theorem

$$A_k = A_n\sigma_k(\alpha_1,\dots,\alpha_n)$$

where σ_k is kth symmetric function of $(\alpha_1,\dots,\alpha_n)$. Then substituting the last expressions for $H_k(A)$ yields by virtue of the homogeneity of H_k that

$$H_k(A) = A_n^{h_k}\, Y_k(\alpha_1,\dots,\alpha_n)$$

where the Y_k, $k=1,2$, are polynomials in α_j. On the other hand, it follows from (37) that $h_1+h_2=n-1$ and the each Y_k must be a divisor of

$$\prod_{i<j}(\alpha_i\alpha_j-1).$$

But the last product consists of irreducible factors $(\alpha_i\alpha_j-1)$ only. Moreover, if one $(\alpha_i\alpha_j-1)$ occures in Y_1 as a divisor then by symmetry the others have to be the divisors as well.

It follows that one of Y_k contains none α_i, i.e. it has the form A_n^p. Thus, applying the remark in the beginning of the proof we see that $p=0$. But this means that Y_k must be a constant factor that contradicts to our assumption and proves the theorem. ■

6. PROOF OF THEOREM 3

Let $P(z)=a_1z+\cdots a_nz^n$, $P\in\mathcal{P}^n_{\mathrm{loc}}$, be a locally injective in the unit disk polynomial. We identify $P'(z)$ with the vector of its coefficients

$$b=(b_1,b_2,\dots,b_n)\in\mathbb{R}^n,$$

where $b_k=ka_k$. We also write $\mathcal{R}(p,q)=\mathcal{R}(P',Q')$ for the corresponding vectors p and q. Moreover, by S we denote the differential operator regarded as a linear transform in $\mathbb{R}^n$:

$$S(P)=P'(z):\mathbb{R}^n\to\mathbb{R}^n.$$

Then the following consequence of (35) and (37)

$$b\in\ker W_{n-1}\Leftrightarrow \mathcal{R}(b,b^*)=0 \tag{40}$$

is useful. Here $b^*=(b_n,\dots,b_1)$ corresponds to P'^*.

LEMMA 2 *The set* $\mathcal{P}^n_{\text{loc}}$ *is an open connected subset of* $\mathbb{R}^n$. *A polynomial* $P(z)$ *is an element of the boundary* $\partial\mathcal{P}^n_{\text{loc}}$ *if and only if the following properties hold*

(i) $P'(z)$ *contains no zeroes in* U;
(ii) $\mathcal{R}(P', P'^{*}) = 0$.

Proof The openness of $\mathcal{P}^n_{\text{loc}}$ obviously follows from the fact that

$$\min_{z\in\overline{U}} |P'(z)| > 0, \quad \forall P \in \mathcal{P}^n_{\text{loc}}. \tag{41}$$

Furthermore, let $P \in \mathcal{P}^n_{\text{loc}}$; then the homotopy

$$a_\lambda = (a_1, a_2t, \dots, a_nt^{n-1}), \quad t \in [0; 1],$$

corresponds to the dilatation $P_t(z) = (1/t)P(tz)$ and connects $P(z)$ and $Q(z) = a_1z$ inside of $\mathcal{P}^n_{\text{loc}}$ since $P'_t(z) = P'(tz) \neq 0$ in $\overline{U}$. In particular, this shows that all the polynomials $P(z)$ with $a_1 > 0$ are contained in a single open component of $\mathcal{P}^n_{\text{loc}}$.

Property (i) easily follows from the continuity arguments and (41).

To prove (ii) we suppose that $P \in \partial\mathcal{P}^n_{\text{loc}}$. Then

$$\min_{z\in\overline{U}} |P'(z)| = 0$$

and it follows from (i) that there is a root ζ_k of $P'(z)$ such that $|\zeta_k| = 1$. On the other hand, the coefficients of P are real and it follows that $\overline{\zeta_k} = 1/\zeta_k$ is a root of P' as well. But this means that $P'(z)$ and $P'^{*}(z)$ has a common root and by the characteristic property of the resultant the latter is equivalent to $\mathcal{R}(P', P'^{*}) = 0$. ■

Proof of Theorem 3 Let us consider a real-valued continuous function

$$f(a) = (-1)^j\mathcal{R}(S(a), S(a)^*) : \ \mathcal{P}_{j,\,\text{loc}} \to \mathbb{R}$$

where the star is used for the corresponding reciprocal polynomial.

We note that f does not change sign on $\mathcal{P}^n_{\text{loc}}$. Indeed, given an arbitrary $P(z) \in \mathcal{P}_{k,\,\text{loc}}$, $k \leq n$, we have that all the roots ζ_j of $P'(z)$ are outside $\overline{U}$. Thus,

$$|\zeta_i\zeta_j| > 1, \quad \forall\, i, j \leq n-1.$$

and by (34) $f(a) \neq 0$. The last inequality together with (40) implies the claimed property. It easily follows from the normalization $a_1 > 0$ and (34) that $f > 0$ on $\mathcal{P}^n_{\text{loc}}$.

Hence, $\mathcal{P}^n_{\text{loc}} \subset \Lambda$ for certain open component Λ of $f > 0$. On the other hand, by property (ii) in Lemma 2 we have $f(a) = 0$ for all $a \in \partial\mathcal{P}^n_{\text{loc}}$. Then applying (40) we get $\Lambda = \mathcal{P}^n_{\text{loc}}$.

Thus, we conclude that $\mathcal{P}^n_{\text{loc}}$ coincides with a certain open component of

$$\mathbb{R}^n \setminus \ker W_{n-1} = \mathbb{R}^n \setminus S^{-1}(\ker f).$$

To finish the proof we only have to check that the three algebraic surfaces in the statement of Theorem 3 have nonempty intersection with the boundary components of $\mathcal{P}^n_{\text{loc}}$ (for $n \geq 3$). Indeed, we notice that the hyperplanes $\Pi^{\pm}$ in (7) correspond to the polynomials $P \in \partial\mathcal{P}^n_{\text{loc}}$ which have their critical points on the real axis: $P'(\pm 1) = 0$. On the other hand, $\mathcal{A}$ in (8) represents the component of $\partial\mathcal{P}^n_{\text{loc}}$ which consists of the polynomials with the complex roots $\zeta \notin \mathbb{R}$, $|\zeta| = 1$, $P'(\zeta) = 0$. The theorem follows. ■

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