Докл. Акад. Наук СССР
Том 314 (1990), № 1

Soviet Math. Dokl.
Vol. 42 (1991), No. 2

# SOME ESTIMATES OF THE MEAN CURVATURE OF GRAPHS OVER DOMAINS IN $\mathbf{R}^n$

UDC 517.95

V. G. TKACHEV

**1.** Let $\Omega \subset \mathbf{R}^n$ be a domain, and let $P$, $Q \subset \overline{\Omega}$ be closed disjoint subsets. The *conformal capacity of the capacitor* $(P, Q; \Omega)$ (see [1], Chapter II, §3) is defined to be

$$\text{(1)} \qquad \operatorname{cap}(P, Q; \Omega) = \inf \int_\Omega \|\nabla \varphi(x)\|^n \, dx, \qquad dx = dx_1 \cdots dx_n,$$

where the infimum is over all possible functions $\varphi(x)$ that are locally Lipschitz on $\Omega$, continuous on $\overline{\Omega}$, and equal to 0 on $Q$ and 1 on $P$. A compact set $P$ is said to have *zero capacity* if there exists a closed set $Q$ with $Q \cap P = \varnothing$ such that $\mathbf{R}^n \setminus Q$ is bounded and $\operatorname{cap}(P, Q; \mathbf{R}^n) = 0$. A closed set $P$ has capacity zero if every compact subset of it does.

Let $H(t)$, $t \in \mathbf{R}$, be a nondecreasing continuous function, and let $f(x) = f(x_1, \dots, x_n)$ be a $C^2$-solution of the equation

$$\text{(2)} \qquad \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \frac{f x_i}{\sqrt{1 + |\nabla f|^2}} \right) = n H(f(x))$$

in the domain $\Omega$. The solutions of this equation are graphs $x_{n+1} = f(x)$ with mean curvature the given function of the coordinate $x_{n+1}$. For $H(t) = a + bt$, $b > 0$, the solutions of (2) describe the phenomenon of capillarity in a column of liquid with cross-section $\Omega$, and has been treated in [2] and [3].

We have the following estimate of the integral mean curvature $H(f(x))$ of the graph $F$ for a solution $x_{n+1} = f(x)$ of (2).

THEOREM 1. *Let $f(x)$ be a solution of* (2), *and let $P \subset \Omega$ be an arbitrary closed set. Then*

$$\text{(3)} \qquad \int_P |H(f(x)|^n dx \le \operatorname{cap}(P, \partial\Omega; \Omega).$$

The idea of the proof consists in the following. We introduce the notation $t^{(n)} = t \cdot |t|^{n-1}$ and fix a function $\varphi(x)$ that is admissible in the variational problem (1) for the capacitor $(P, \partial\Omega; \Omega)$. We consider an arbitrary $C^1$-smooth function $H_1(t)$ with $H_1'(t) \ge 0$ such that

$$\text{(4)} \qquad |H_1(t)| \le |H(t)|, \;\; H(t) \cdot H_1(t) \ge 0.$$

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53A10, 76B45; Secondary 31B15, 53A05.

Using (2), we arrive at the inequality

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[ H_1^{(n-1)}(f(x))\varphi^n(x) \frac{f_{x_i}}{\sqrt{1+|\nabla|^2}} \right]$$
$$\geq nH_1^{(n-1)}(f(x))H(f(x))\varphi^n(x) + nH_1^{(n-1)}(f(x))\varphi^{n-1}(x) \sum_{i=1}^{n} \frac{\varphi_{x_i} f_{x_i}}{\sqrt{1+|\nabla f|^2}}.$$

First using (4), we integrate the last inequality and use the Cauchy formula. As a result,

$$\int_\Omega \varphi^n(x)|H_1(f(x)|^n dx \leq \int_\Omega \|\nabla_\varphi\|^n \, dx.$$

Using the condition $\varphi \equiv 1$ on $\varphi(x)$ for $x \in P$, we pass to the infimum with respect to $\varphi(x)$ in the inequality just obtained:

$$\int_P |H_1(f(x))|^n \, dx \leq \operatorname{cap}(P, \partial\Omega; \Omega).$$

Finally, approximating $H(t)$ by functions $H_1(t)$ satisfying (4), we arrive at the required estimate (3).

COROLLARY 1. *Let $f(x)$ be the solution of (2), which is defined everywhere in $\mathbf{R}^n$ except perhaps for a closed set $P$ of capacity zero. Then the mean curvature satisfies $H(f(x)) \equiv 0$ everywhere in $\mathbf{R}^n \setminus P$. In particular:*
a) *If $2 \leq n \leq 7$, then $f(x)$ is a linear function.*
b) *If $H(t)$ is strictly monotone, then $f(x) \equiv \text{const}$ for $n \geq 2$.*

In the two-dimensional case, if $H(t)$ is of constant sign, $H'(t) \geq 0$, and the set $P$ is empty, assertion a) is due to Cheng and Yau [4].

**2.** NOTATION. $B(x, R)$ is the ball of radius $R > 0$ about $x \in \mathbf{R}^n$, and $\operatorname{dist}(x, E)$ is the distance from a point $x$ to a set $E$.

THEOREM 2. *Let $f(x)$ be a solution of equation (2) in the domain $\Omega \subset \mathbf{R}^n$. Then*

$$\sup_{x\in\Omega}\{|H(f(x))| \cdot \operatorname{dist}(x, \partial\Omega)\} \leq 1. \tag{5}$$

*Equality is attained in (5) in the case when $\Omega$ is a ball and $f(x)$ describes a hemisphere over $\Omega$.*

COROLLARY 2. *Let $f(x)$ be a solution of (2) in the ball $B(0, \mathbf{R})$. Then*

$$|H(f(0))| \leq 1/R.$$

This assertion was obtained by Bernstein [5] and Finn [3] in the cases of mean curvature that is constant or bounded away from zero, respectively.

It is not hard to see that there exists a bounded radially symmetric $C^2$-solution $f(x)$ of equation (2) with right-hand side

$$H(f(x)) = \frac{n-1}{n}(R^n - \|x\|^n)^{-1/n}, \qquad x \in B(0; R),$$

i.e., the mean curvature $H(f(x))$ increases without bound in a neighborhood of any point of the boundary $\partial B(0, R)$. It is obvious from the following assertion that such boundary singularities cannot be isolated. Namely, we have

THEOREM 3. *Suppose that* $\Omega \subset \mathbf{R}^n$ *is a domain and* $q \in \Omega$ *is a point. Let* $f(x)$ *be a solution of* (2) *in* $\Omega \backslash \{q\}$. *Then the function* $H(f(x))$ *is bounded in a neighborhood of* $q$, *and*

$$\overline{\lim_{x \to q}} |H(f(x))| \leq (\operatorname{dist}(q\,;\,\partial\Omega))^{-1}.$$

**3.** It is interesting to determine the exact value of the functional on the left-hand side of (5) in the case of an arbitrary domain $\Omega$ different from a ball. Below we give examples of domains for which this problem can be completely solved.

For any arbitrary integer $1 \leq p \leq n$ let $\Pi_{n,p}$ be a layer in space that, to within a motion and a homothety, is the coordinate product

$$\Pi_{n,p} = D^p(a) \times \gamma^{n-p}, \tag{6}$$

where $\gamma^{n-p}$ is an $(n-p)$-dimensional plane, and $D^p(a)$ is a $p$-dimensional disk of radius $a > 0$.

THEOREM 4. *Let* $\Omega$ *coincide with one of the domains of the form* (6). *Then*

$$\sup_{x \in \Omega}\{|H(f(x))| \cdot \operatorname{dist}(x\,,\,\partial\Omega)\} \leq \frac{p}{n} \tag{7}$$

*for any solution* $f(x)$ *of equation* (2) *in* $\Omega$. *Equality is attained in cases of special surfaces of constant mean curvature over* $\Omega$.

We sketch the proof. Let $\Omega = Pi_{n,p}$, $1 \leq p \leq n-1$. Using the properties of the mean curvature of a hypersurface in $\mathbf{R}^{n+1}$, we can assume without loss of generality that $\Pi_{n,p}$ has the form

$$\pi_{n,p} = \left\{ x \in \mathbf{R}^{n+1} : \sum_{i=1}^{p} x_i^2 < 1\,;\; x_{n+1} = 0 \right\},$$

and it suffices to establish (7) for $x = 0$.

Let $v$ and $w$ denote the projections of a vector $x \in \mathbf{R}^{n+1}$ on the following mutually orthogonal subspaces of $R^{n+1}$: $\gamma = \{x \in \mathbf{R}^{n+1} : x_i = 0;\; 1 \leq i \leq p\}$ and $\gamma^{\perp}$, respectively. Consider the $\lambda$-parametric family of tori

$$T_\lambda(R) = \{x = (w\,;\,v) \in \mathbf{R}^{n+1} : (\|v - \lambda e_{n+1}\| - R)^2 + \|w\|^2 = r^2\},$$

where $0 < r < 1 < R$ are fixed numbers, and $e_{n+1}$ is a coordinate vector. It is clear that $T_\lambda(r)$ does not intersect the surface $F$ for sufficiently large numbers $\lambda > 0$. We find the infimum $\lambda_0$ of such numbers $\lambda$. Then $T_{\lambda_0}(R)$ is everywhere not below the $F$, and touches it at the same point $x_0 \in \Pi_{n,p}$. Comparing the mean curvature of the torus $H_T$ at the point $x_0$, we arrive at the inequality

$$H(f(x_0)) \leq H_T(x_0) = \frac{1}{r}\left(\frac{p}{n} + \frac{n-p}{n}\frac{\|v_0\| - R}{\|v_0\|}\right)$$

for the mean curvature of the surface, where $x_0 = (w_0\,,\,v_0)$. But the point $(0\,;\,f(0))$ of $F$ lies no higher than the point $(x_0\,;\,f(x_0))$, and hence, since $H(t)$ is monotone,

$$H(f(x_0)) \leq \frac{1}{r}\left(\frac{p}{n} + \frac{n-p}{n}\frac{1}{R-1}\right).$$

The required estimate is obtained by passing to the limit as $r \to 1$ and $R \to \infty$. A lower estimate is established similarly. The case of equality holds, for example, when

$$f(x) = \sqrt{1 - x_1^2 - \cdots - x_p^2},$$

with $H \equiv p/n$.

The author thanks V. M. Miklyukov for his interest in this work, and for useful discussions.

Volgograd State University Received 21/JUNE/89

BIBLIOGRAPHY

1. Yu. G. Reshetnyak, *Space mappings with bounded distortion*, "Nauka", Novosibirsk, 1982; rev. English transl., Amer. Math. Soc., Providence, R.I., 1989.

2. Robert Finn, J. Analyse Math. **14** (1965), 139–160.

3. ——, Z. Angew. Math. Mech. **61** (1981), 165–173, 175–177.

4. S. Y. Cheng and S. T. Yau, Comm. Pure Appl. Math. **28** (1975), 333–354.

5. S. N. Bernstein, Ann. Sci. École Norm. Sup. (3) **27** (1910), 233–256.

Translated by H. H. McFADEN