

A GENERALIZATION OF CARTAN'S THEOREM ON ISOPARAMETRIC CUBICS

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ABSTRACT. We generalize the well-known result of É. Cartan on isoparametric cubics by showing that a homogeneous cubic polynomial solution of the eiconal equation $|\nabla f|^2 = 9|x|^4$ must be rotationally equivalent to either $x_n^3 - 3x_n(x_1^2 + \dots + x_{n-1}^2)$ or to one of four exceptional Cartan cubic polynomials in dimensions $n = 5, 8, 14, 26$.

1. INTRODUCTION

In his paper [1] É. Cartan found all cubic homogeneous polynomials in \mathbb{R}^n , $n \geq 3$, satisfying the isoparametric equations

$$(1.1) \quad |\nabla f(x)|^2 = 9|x|^4,$$

$$(1.2) \quad \Delta f(x) = 0.$$

Amazingly, the cubic solutions of (1.1)-(1.2) can be described by means of four real division algebras \mathbb{F}_d of dimension d , where $\mathbb{F}_1 = \mathbb{R}$ (reals), $\mathbb{F}_2 = \mathbb{C}$ (complexes), $\mathbb{F}_4 = \mathbb{H}$ (quaternions) and $\mathbb{F}_8 = \mathbb{O}$ (octonians). Cartan proved that for a cubic solution to exist the dimension n must be 5, 8, 14 or 26, i.e.

$$n = 3d + 2, \quad d = 1, 2, 4, 8,$$

and, in this case, the solution is congruent (i.e. rotationally equivalent) to one of the following polynomials (cf. [1, p. 34]):

$$f_d(x) = x_n^3 - 3x_n x_{n-1}^2 + \frac{3}{2}x_n(X_0\bar{X}_0 + X_1\bar{X}_1 - 2X_2\bar{X}_2) \\
 + \frac{3\sqrt{3}}{2}x_{n-1}(X_0\bar{X}_0 - X_1\bar{X}_1) + \frac{3\sqrt{3}}{2}((X_0X_1)X_2 + \bar{X}_2(\bar{X}_1\bar{X}_0)),$$

where $x = (X_0, X_1, X_2, x_{n-1}, x_n)$ and vector $X_k = (x_{kd+1}, \dots, x_{kd+d})$ are identified with the corresponding elements of \mathbb{F}_d , $k = 0, 1, 2$, and \bar{X} denotes the conjugate of X in \mathbb{F}_d . It is not hard to prove also that all the Cartan polynomials are irreducible.

In dimension $n = 2$ there is also a *reducible* polynomial satisfying (1.1)-(1.2),

$$f_0(x) = x_2^3 - 3x_2x_1^2 = \operatorname{Re}(x_2 + x_1\sqrt{-1})^3,$$

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which, though having no evident relation to the division algebras, can be thought of (at least formally) as the member of the above family corresponding to $d = 0$, where all X_i are supposed to be zero. It is easy to see that in higher dimensions $n \geq 3$, this polynomial $f_0(x)$ gives rise to a new family of (reducible) solutions of the eiconal equation (1.1) *alone*, namely

$$(1.3) \quad f_0(x) = x_n^3 - 3x_n(x_1^2 + \cdots + x_{n-1}^2).$$

Note also that f_0 is not congruent to any of the f_d in the corresponding dimensions because all f_d are harmonic, while $\Delta f_0 = 6(2-n)x_n \neq 0$ for $n \geq 3$.

Our main result is the following characterization of cubic solutions of equation (1.1) alone.

Theorem 1.1. *Any homogeneous cubic polynomial satisfying the eiconal equation $|\nabla f|^2 = 9|x|^4$ is rotationally equivalent to either $x_n^3 - 3x_n(x_1^2 + \cdots + x_{n-1}^2)$ or to one of the exceptional Cartan cubic polynomials $f_d(x)$ in dimensions $n = 5, 8, 14, 26$. In particular, irreducible cubic solutions of (1.1) can exist only in dimensions $n = 5, 8, 14$ and 26 .*

Remark 1.2. It is well known that for $d \neq 0$ the focal varieties $f_d = 0$ are minimal cones in \mathbb{R}^{3d+2} (i.e. immersed submanifolds having zero mean curvature). So far, these four Cartan cones are the only known examples of minimal cubics besides the cubic $2x_1x_2x_3 + (x_1^2 - x_2^2)x_4$ (a member of Lawson's family of algebraic minimal surfaces in \mathbb{R}^4 given in [3]) and two additional cubics, each in dimensions 9 and 15, found by Wu-yi Hsiang in [2]. In a forthcoming paper [5] we provide a classification of minimal cubics in \mathbb{R}^n , and Theorem 1.1 above plays a crucial role in constructing the so-called exceptional family of minimal cubics in \mathbb{R}^{3k} .

2. SYMMETRIC COMPOSITION FORMULAS

Recall that a composition formula of size $[r, s, m]$ over the field of real numbers (see [4]) is an identity

$$\sum_{k=1}^m b_k^2(x, y) = |x|^2|y|^2, \quad x \in \mathbb{R}^r, y \in \mathbb{R}^s,$$

where $b_k(x, y)$ are real bilinear forms and $|x|^2 = \langle x, x \rangle$ is the usual Euclidean norm of x . It is well known that the existence of a composition formula of size $[r, s, m]$ is equivalent to solvability of the Hurwitz matrix equations

$$(2.1) \quad A_i^t A_i = 1_s, \quad 1 \leq i \leq r,$$

$$(2.2) \quad A_i^t A_j + A_j^t A_i = 0, \quad i \neq j$$

(see, for instance, [4]). Here $A_i \in \mathbb{R}^{m \times s}$ is a matrix of size $m \times s$ with real entries, A^t denote the transpose matrix, and 1_k stands for the unit matrix in $\mathbb{R}^{k \times k}$. It follows from (2.1) that $m \geq \max\{r, s\}$.

If $\max\{r, s\} = m$, say $s = m$, then the celebrated Hurwitz-Radon theorem states that a composition formula of size $[r, m, m]$ exists (equivalently, the Hurwitz matrix system of size $[r, m, m]$ is solvable) iff

$$r \leq \rho(m),$$

where the Hurwitz-Radon function $\rho(m)$ is defined for positive integers $m \geq 1$ by the formula

$$(2.3) \quad \rho(m) = 8a + 2^b, \quad \text{where } m = 2^{4a+b} \cdot \text{odd}, \quad 0 \leq b \leq 3,$$

and $\rho(m) = 0$ otherwise. In particular, for positive integers we always have $\rho(m) \leq m$ with equality only if $m = 1, 2, 4, 8$. Another useful observation is that $\rho(m) = 1$ if and only if m is odd.

We shall need an analogue of the Hurwitz-Radon function for symmetric solutions of (2.1)-(2.2). Given $m \geq 1$ we define $\rho_{\text{symm}}(m)$ as the maximal possible r such that the Hurwitz matrix equations are solvable for symmetric matrices $A_i \in \mathbb{R}^{m \times m}$, $i = 1, \dots, r$.

Proposition 2.1. *For any $m \geq 1$,*

$$(2.4) \quad \rho_{\text{symm}}(m) = 1 + \rho\left(\frac{m}{2}\right).$$

Moreover, if $\{A_i\}_{1 \leq i \leq r}$ is a symmetric solution of (2.1)-(2.2) for $r = \rho_{\text{symm}}(m) \geq 2$, then all the matrices are trace free: $\text{trace } A_i = 0$.

Proof. First suppose that $\rho_{\text{symm}}(m) = 1$. Then m must be an odd number, because otherwise $m = 2k$, $k \in \mathbb{Z}$, and the two matrices

$$A_1 = \begin{pmatrix} -1_k & 0 \\ 0 & 1_k \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1_k \\ 1_k & 0 \end{pmatrix}$$

provide a symmetric solution of (2.1)-(2.2) with $r = 2$. Thus m is odd and it follows from (2.3) that $\rho(m) = 1$. This proves (2.4) for $\rho_{\text{symm}}(m) = 1$.

Now let us consider the case $r := \rho_{\text{symm}}(m) \geq 2$. Then we can find a symmetric solution $\{A_i\}_{1 \leq i \leq r}$ of (2.1)-(2.2). Without loss of generality we can assume that A_r has the diagonal form, say $A_r = \text{diag}(a_1, \dots, a_m)$, where $a_i \in \mathbb{R}$. Then (2.1) implies that $a_i^2 = 1$; that is, after a suitable rotation we get

$$(2.5) \quad A_r = 1_t \oplus (-1)_{m-t}, \quad 0 \leq t \leq m.$$

We claim that $t(m-t) \neq 0$. Indeed, if $t = 0$ or $t = m$, then $A_r = \pm 1_m$; hence applying (2.2) to A_i and A_r we find that $A_i^t + A_i = 0$, $1 \leq i \leq r-1$, which in turn implies that $A_i = 0$ because the A_i are symmetric. But the latter contradicts (2.1) for $r \geq 2$. Hence A_r has eigenvalues of both signs, i.e. $1 \leq t \leq m-1$ in (2.5).

Write the remaining A_i in the block form associated with the polarization of \mathbb{R}^m given by (2.5),

$$A_i = \begin{pmatrix} C_i & E_i \\ E_i^t & D_i \end{pmatrix}, \quad 1 \leq i \leq r-1.$$

Here $C_i \in \mathbb{R}^{t \times t}$ and $D_i \in \mathbb{R}^{(m-t) \times (m-t)}$ are symmetric matrices, and $E_i \in \mathbb{R}^{t \times (m-t)}$. Applying again (2.2) to A_i and A_r we find immediately that C_i and D_i are zero matrices for $i \leq r-1$. Furthermore, (2.1) yields

$$(2.6) \quad E_i E_i^t = 1_t, \quad E_i^t E_i = 1_{m-t},$$

and setting $1 \leq i, j \leq r-1$ in (2.2) we get

$$(2.7) \quad E_i E_j^t + E_j E_i^t = 0.$$

Now observe that identity $E_i E_i^t = 1_t$ implies $m-t \geq t$, and similarly, $E_i^t E_i = 1_{m-t}$ implies $t \geq m-t$. Hence $m = 2t$; in particular, m is an even number. It follows that all E_i are quadratic matrices and equations (2.6)-(2.7) are equivalent

to the Hurwitz matrix equations of size $[r-1, t, m-t] \equiv [r-1, \frac{m}{2}, \frac{m}{2}]$. This implies by the definition of ρ that $r-1 \leq \rho(m/2)$, i.e.

$$\rho_{\text{symm}}(m) - 1 \leq \rho(m/2).$$

In order to prove the inverse inequality, let us fix an even $m \geq 2$ and set

$$r := \rho(m/2) + 1 \geq 2.$$

Let $\{E_i\}_{1 \leq i \leq r}$ be an arbitrary solution of (2.1)-(2.2) of size $[r-1, \frac{m}{2}, \frac{m}{2}]$. Then it is easy to check that the symmetric matrices

$$A_i = \begin{pmatrix} 0 & E_i \\ E_i^t & 0 \end{pmatrix}, \quad 1 \leq i \leq r-1, \quad A_r = 1_{m/2} \oplus (-1)_{m/2}$$

give a solution to (2.6)-(2.7) of size $[r, m, m]$. Thus $\rho_{\text{symm}}(m) \geq r = \rho(m/2) + 1$, which finishes the proof of (2.4).

The last statement of the proposition easily follows from the block form of A_i and the fact that trace is invariant with respect to orthogonal transformations. \square

3. PROOF OF THEOREM 1.1

Let f be any cubic polynomial satisfying (1.1). Then $f \not\equiv 0$ and it can be brought into the normal form, i.e.

$$(3.1) \quad f(x) = x_n^3 + 3x_n A(\bar{x}) + 3B(\bar{x}), \quad \bar{x} = (x_1, \dots, x_{n-1}),$$

where A is a quadratic form and B is a cubic form in \bar{x} . Indeed, the maximum value of $f(x)$ on the unit sphere $|x| = 1$ is strictly positive and attained at some point x^0 . Then $\nabla f(x^0) = cx^0$; hence by homogeneity of f ,

$$c = \langle x^0, \nabla f(x^0) \rangle = 3f(x^0) \neq 0,$$

and it is easily shown that in the new orthogonal coordinates with x_0 being the n th vector, f takes the form (3.1).

Equating $|\nabla f|^2$ to $9|x|^2$ yields

$$x_n^2(2A + |\nabla A|^2 - 2|\bar{x}|^2) + 2x_n \langle \nabla A, \nabla B \rangle + (A^2 + |\nabla B|^2 - |\bar{x}|^4) = 0,$$

where $\bar{x} = (x_1, \dots, x_{n-1})$. Thus

$$(3.2) \quad 2A + |\nabla A|^2 - 2|\bar{x}|^2 = 0,$$

$$(3.3) \quad \langle \nabla A, \nabla B \rangle = 0,$$

$$(3.4) \quad A^2 + |\nabla B|^2 = |\bar{x}|^4.$$

We can assume without loss of generality that A is given in the diagonal form, say $A(\bar{x}) = \text{diag}(a_1, \dots, a_{n-1})$, so that (3.2) yields $2a_i^2 + a_i - 1 = 0$. This implies that a_i is either $\frac{1}{2}$ or -1 . We redenote the coordinates such that

$$(3.5) \quad A(\bar{x}) = \frac{1}{2} \sum_{i=1}^p \xi_i^2 - \sum_{j=1}^q \eta_j^2, \quad p+q = n-1, \quad \bar{x} = (\xi, \eta).$$

Denote by V_1 and V_2 the corresponding eigenspaces of dimensions p and q , respectively. The thus obtained polarization $V \equiv \mathbb{R}^{n-1} = V_1 \oplus V_2$ induces the corresponding decompositions in the tensor products, in particular,

$$V^{*\otimes 3} \simeq V^{3,0} \oplus V^{2,1} \oplus V^{1,2} \oplus V^{0,3}, \quad V^{i,j} = V_1^{*\otimes i} \otimes V_2^{*\otimes(3-i)}.$$

According to the latter decomposition we have for the cubic form B ,

$$B = B_{3,0} + B_{2,1} + B_{1,2} + B_{0,3},$$

where $B_{i,3-i} \equiv B_{i,3-i}(\xi, \eta) \in V^{i,3-i}$ are linearly independent cubic forms. By homogeneity one finds

$$\begin{aligned} \langle \xi, \nabla B_{i,3-i} \rangle &= iB_{i,3-i}, \\ \langle \eta, \nabla B_{i,3-i} \rangle &= (3-i)B_{i,3-i}; \end{aligned}$$

hence by virtue of (3.3),

$$\langle \nabla A, \nabla B \rangle = \langle \xi - 2\eta, \sum_{i=0}^3 \nabla B_{i,3-i} \rangle = -6B_{3,0} - 3B_{1,2} + 3B_{0,3} = 0.$$

It follows from the linear independence of $B_{i,3-i}$ that $B_{3,0} = B_{1,2} = B_{0,3} = 0$. Thus $B \in V^{2,1}$, i.e.

$$(3.6) \quad B \equiv B_{2,1} = \sum_{i=1}^q \eta_i Q_i(\xi),$$

where $Q_i(\xi) \in V_2^{\otimes *2}$ is a quadratic form in ξ .

Note that $q \geq 1$, since otherwise we would have $B \equiv 0$ and by virtue of (3.4), $A^2 = |\bar{x}|^4$, which would imply a contradiction to (3.5), because $A = \frac{1}{2}|\bar{\xi}|^2 \equiv \frac{1}{2}|\bar{x}|^2$ for $q = 0$. Thus $q = \dim V_2 \geq 1$.

Note also that if $\dim V_1 = p = 0$, then $B = 0$ and $A = -|\eta|^2$. It is easy to check that (3.2)-(3.4) turn into identities and the corresponding f becomes the solution of (1.1) in the form (1.3).

It only remains to treat the case when both V_1 and V_2 are nontrivial: $\dim V_k \geq 1$, $k = 1, 2$. We have from (3.4),

$$\left(\frac{1}{2}|\xi|^2 - |\eta|^2 \right)^2 + \sum_{i=1}^q Q_i^2(\xi) + \left| \sum_{i=1}^q \eta_i \nabla Q_i(\xi) \right|^2 = (|\xi|^2 + |\eta|^2)^2.$$

Regarding the latter equality as an identity in $\mathbb{R}[\eta_1, \dots, \eta_q]$, one finds

$$(3.7) \quad \sum_{i=1}^q Q_i^2(\xi) = \frac{3}{4}|\xi|^4$$

and

$$(3.8) \quad \langle \nabla Q_i, \nabla Q_j \rangle = 3\delta_{ij}|\xi|^2,$$

where δ_{ij} is the Kronecker delta. Write Q_i in the matrix form

$$(3.9) \quad Q_i(\xi) = \frac{\sqrt{2}}{3}\xi^t A_i \xi,$$

where $A_i \in \mathbb{R}^{p \times p}$ is symmetric. It follows then from (3.8) that the symmetric matrices $\{A_i\}_{1 \leq i \leq q}$ solve the Hurwitz matrix equations (2.1)-(2.2) for $s = m = p$ and $r = q$; therefore

$$(3.10) \quad q \leq \rho_{\text{symm}}(p).$$

If $q = 1$, then (3.7)-(3.8) immediately yields $Q_1 = \frac{\sqrt{2}}{3}|\xi|^2$ (the choice of sign of Q_1 is immaterial because we are free to change the sign of x_n in (3.1)). Thus

$$f(x) = x_n^3 + \frac{3}{2}x_n(x_1^2 + \dots + x_{n-2}^2 - 2x_{n-1}^2) + 3\sqrt{3}x_{n-1}(x_1^2 + \dots + x_{n-2}^2).$$

But the latter polynomial is exactly the solution (1.3) after a suitable rotation, namely,

$$f(x_1, x_2, \dots, x_{n-1}, x_n) = f_0(x_1, x_2, \dots, \frac{\sqrt{3}}{2}x_{n-1} + \frac{1}{2}x_n, -\frac{1}{2}x_n - \frac{\sqrt{3}}{2}x_{n-1}).$$

Finally, let us suppose that $q \geq 2$. Then (3.7) means that

$$(3.11) \quad y(\zeta) := \frac{2}{\sqrt{3}}(Q_1(\zeta), \dots, Q_q(\zeta)) : S^{p-1} \rightarrow S^{q-1}$$

is a quadratic map sending the unit sphere $|\zeta| = 1$ to the unit sphere $|y| = 1$. Note also that our assumption $q \geq 2$ implies by virtue of (3.8) that the image of $y(S^{p-1})$ in S^{q-1} is distinct from a point. Then one result of P. Yiu [6] provides an obstruction for a nonconstant quadratic map to exist if the dimension $q - 1$ of the target sphere is too small. More specifically, let us denote by $\sigma(k)$, $k \geq 1$, the minimal possible value of l for which there exists a *nonconstant* homogeneous quadratic map $S^k \rightarrow S^l$. Then the theorem of P. Yiu [6, Theorem 4] (see also [7] for general polynomial maps) yields a recursive formula for $\sigma(k)$:

$$\sigma(2^a + b) = \begin{cases} 2^a, & 0 \leq b < \rho(2^a), \\ 2^a + \sigma(b), & \rho(2^a) \leq b < 2^a. \end{cases}$$

We shall need only two easy consequences of the Yiu formula, namely, that $\sigma(m)$ is a nondecreasing function on \mathbb{Z}^+ and

$$(3.12) \quad \sigma(2^a) = 2^a, \quad a \in \mathbb{Z}^+.$$

In this set-up, one can rewrite the existence of a nonconstant quadratic map (3.11) as the lower estimate

$$q - 1 \geq \sigma(p - 1).$$

Combining this with (3.10), we get after applying Proposition 2.1 that

$$(3.13) \quad 1 + \sigma(p - 1) \leq q \leq 1 + \rho\left(\frac{p}{2}\right).$$

By our assumption $q \geq 2$; hence the right inequality in (3.13) implies that $\rho\left(\frac{p}{2}\right) \geq 1$, i.e. p is even. We write this as $p = 2^{\nu+1}p_0$, where p_0 is an odd number and $\nu \geq 0$. Then (3.13) and the definition of ρ yield that

$$(3.14) \quad 1 + \sigma(2^{\nu+1}p_0 - 1) \leq q \leq 1 + \rho(2^\nu).$$

Notice first that $p_0 = 1$, because otherwise we would have $p_0 \geq 3$ and by monotonicity of σ ,

$$\sigma(2^{\nu+1}p_0 - 1) \geq \sigma(3 \cdot 2^{\nu+1} - 1) \geq \sigma(2^{\nu+2}) = 2^{\nu+1}.$$

But the latter contradicts the right inequality in (3.14) in view of $\rho(2^\nu) \leq 2^\nu$. Thus $p_0 = 1$ and we rewrite (3.14) as

$$1 + \sigma(2^{\nu+1} - 1) \leq q \leq 1 + \rho(2^\nu).$$

Now applying $\sigma(2^{\nu+1} - 1) \geq \sigma(2^\nu) = 2^\nu$ and $\rho(2^\nu) \leq 2^\nu$, we find

$$1 + 2^\nu \leq q \leq 1 + 2^\nu,$$

where the right inequality is strong for $\nu \geq 4$. Thus the only possible values are $q = 2^\nu + 1$, $\nu = 0, 1, 2, 3$, and an easy check shows that all the values are compatible with (3.13). The corresponding values of p , q and the resulting dimension n are displayed in Table 1.

TABLE 1. Exceptional values of p and q

ν	$p = 2^{\nu+1}$	$q = 2^\nu + 1$	$n = p + q + 1$
0	2	2	5
1	4	3	8
2	8	5	14
3	16	9	26

One can see that the last column contains exactly the dimensions of the isoparametric cubics found by É. Cartan that were mentioned in the Introduction. To finish the proof, it suffices only to show that for the values of p and q as in Table 1, any solution f of (1.1) is harmonic. But this is true because $q = 2^\nu + 1$ and $p = 2^{\nu+1}$; hence from (3.5) we get $\Delta A(\bar{x}) = p - 2q = -2$, and by virtue of (3.1) and (3.6), we find

$$\Delta f = 3(2 + \Delta A(\bar{x}))x_n + 3\Delta B(\bar{x}) = \sum_{i=1}^q \eta_i \Delta Q_i(\xi).$$

On the other hand, for $q \geq 2$ the matrices A_i in (3.9) by Proposition 2.1 are trace free, so that $\Delta Q_i = 2 \text{trace} Q_i = 0$. Thus, $\Delta f = 0$ and we get (1.1)-(1.2). Applying Cartan's theorem finishes the proof.

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