ISSN 0197-7156

SOVIET MATHEMATICS (Iz. VUZ)

(Izvestiya VUZ. Matematika)

Vol. 31, No. 7

ALLERTON PRESS INC.

STRUCTURE IN THE LARGE OF EXTERNALLY COMPLETE MINIMAL SURFACES IN R^3

V. M. Miklyukov and V. G. Tkachev

Izvestiya VUZ. Matematika,
Vol. 31, No. 7, pp. 30-36, 1987

UDC 519.3

1. Let M be a two-dimensional noncompact oriented manifold. Consider the surface (M, x) specified by the C^2-immersion $x: M \to R^3$. It is assumed everywhere below that the immersion is proper, i.e., the preimage of any compact set $F \subset R^3$ is a compact. This means that surface (M, x) is externally complete and the closure $\overline{x(M)}$ of set x(M) in R^3 coincides with x(M).

Surface (M, x) is termed minimal if its mean curvature exactly vanishes.

We introduce the following concept. Let V be a plane in R^3. For an arbitrary point $v \in V$, we denote by the symbol N(v) the number of points of intersection (taking their multiplicity into account) between set x(M) and the straight line passing through point v orthogonally to plane V. We will assume that, for all $v \in V$, $N(v) < \infty$ is satisfied. It is easily established that function N(v) is Lebesgue-measurable and the quantity

$$n(t, V) = \int_{|v|=t} N(v) |dv|.$$

is determined for almost all t > 0.

The purpose of the present paper is to prove the following assertion reflecting certain specific features of the structure of surface (M, x) in the large.

Theorem 1. Let (M, x) be a minimal surface. If

$$x(M) \cap V = \varnothing$$

and

$$\lim_{t \to \infty} \frac{n(t, V)}{t \ln t} = 0, \tag{1}$$

are satisfied for some plane $V \subset R^3$, then (M, x) is a plane.

This theorem is also of definite interest in connection with Calabi's problem of the existence of nontrivial (i.e., different from the plane) complete minimal surfaces in R^3 located in a half-space. The absence of globally minimal nontrivial surfaces lying in a half-space was proved by Miranda [1]. The absence in R^3 of minimal surfaces tubular in the large located in a half-space follows from the results of Miklyukov [2]. An example of an internally complete nontrivial minimal surface enclosed between two parallel planes in R^3 was constructed by Jorge and Xavier [3].

It follows from theorem 1, that for nontrivial minimal surfaces, the "mean" number of points lying over the circle $|v| = t$ grows rather quickly. The same is also true in the other method described below for calculating the multiplicity of the number of points. Let $S(t) = S(x_0, t)$ be a sphere of radius $t > 0$ centered at the point $x_0 \in R^3$, $S(1) = S$ and let $\tau \in S$ be an arbitrary point. We denote by $l(\tau)$ a ray starting at point x_0 and passing through point τ and by $N_1(\tau, t)$ the number of points of intersection between surface (M, x) and ray $l(\tau)$ that lie in ball $B(x_0; t) = \{x \in R^3 : |x - x_0| < t\}$. Let $N_1(\tau, t) < \infty$ for all $\tau \in S$ and all $t > 0$. Assume

$$n(t, S) = \int_S N_1(\tau, t)\, d\sigma_\tau,$$

where $d\sigma_\tau$ is an area element on sphere S.

There holds

Theorem 2. Let (M, x) be a minimal surface in R^3. Assume that, for some point $x_0 \in R^3$,

$$\lim_{t \to \infty} \frac{n(t, s)}{\ln t} = 0, \tag{2}$$

is satisfied. Then, if surface (M, x) is located in a half-space, (M, x) is a plane.

The essential part of the proof of theorem 2 consists in estimating the growth rate of the "logarithmic area" of the set $x(M) \cap B(x_0, t)$. A somewhat weaker result is obtained if, instead of the logarithmic area, one estimates the area of this set or that of a geodesic disk on surface (M, x), as is done by Cheng and Yau [4]. The proof of theorem 1 is more specialized.

2. Consider the surface (M, x) specified by the C^2-immersion $x(m): M \to R^3$. The standard metric induces on R^3 under the mapping x(M) the Riemannian metric and a connection on manifold M. We denote by $\langle a, b \rangle$ the scalar product of the vectors in the tangent bundle.

We will employ the modulus-capacity technique elaborated in [5]. The requisite concepts will be briefly introduced here. Let $P, Q \in M$ be disjoint closed sets and $\phi(m)$ be an arbitrary locally Lipschitz function reverting to 0 and 1 on sets P and Q respectively. We will determine the capacity of condenser (P, Q), assuming

$$\operatorname{cap}(P, Q) = \inf \int_M |\nabla\varphi|^2,$$

where the symbol $\nabla\phi$ denotes the vector-gradient ϕ in the metric of manifold M and $|\nabla\varphi|^2 = \langle\nabla\varphi, \nabla\varphi\rangle$.

We will say that surface (M, x) has parabolic conformal type if, for any compact $F \subset M$ there exists the exhaustion $A_k \subset A_{k+1}, \bigcup_{k=1}^{\infty} A_k = M$ of manifold M successively covered by sets $A_k \supset F$ with compact closures, for which

$$\lim_{k\to\infty} \operatorname{cap}(F, M \setminus A_k) = 0. \quad (3)$$

We will utilize the following concept in addition to capacity. Let Γ be a family of locally rectifiable arcs γ in M. We will say that the Lebesgue-measurable function $\rho(m) \geq 0$ locally bounded in essence is admissible for family Γ if, for any arc $\gamma \in \Gamma$,

$$\int_\gamma \rho(m) > 1.$$

is satisfied. The quantity

$$\operatorname{mod} \Gamma = \inf \int_M \rho^2(m),$$

with the exact lower bound taken over all possible functions $\rho(m)$ admissible for family Γ, is called the modulus of family Γ.

There holds

Lemma 1. Let (P, Q) be an arbitrary condenser and Γ be a family of locally rectifiable arcs in M joining sets P and Q. Then,

$$\operatorname{cap}(P, Q) = \operatorname{mod} \Gamma. \quad (4)$$

Proof. The proof basically follows that of the corresponding Fuglede assertion [6] for condensers in R^n, and we will consequently omit it. Individual details related to the need to conduct the argument in the general Riemannian metric will be found in [5], where an analogous assertion was proved for condensers located on the graphs of Lipschitz functions.

3. The following statement is supplemental in the present study, but it is of interest in itself as a test for parabolicity of conformal type in a minimal surface.

Lemma 2. Let (M, x) be a minimal surface in R^3. If this surface has property (1) or property (2), then it is of parabolic conformal type.

We will first prove a simpler case, i.e., that in which surface (M, x) is assumed to satisfy condition (2). It can be presumed with no loss of generality that $x_0 = x$. Let $F \subset M$ be a compact set. Specify $r > 0$ so that set x(F) is contained in the ball B(0, r). Arbitrarily chose $R > r$ and denote by A_R the set $x(M) \cap B(0, R)$. Assume $A_R^* = x^{-1}(A_R)$. It is clear that A_R^* is an open subset of M having a compact closure.

If $R_1 < R_2 < \ldots$ and $\lim_{k\to\infty} R_k = \infty$, the sequence of sets $A_{R_1}^* \subset A_{R_2}^* \subset \ldots$ forms the exhaustion of manifold M. In order to show that this exhaustion has property (3), we will consider the family Γ_R of locally rectifiable arcs $\gamma \in A_R^*$, joining compact G to set $M \setminus A_R^*$. By equality (4), it is sufficient to show that the lower bound of the modulus of family Γ_R equals zero as $R \to \infty$.

Let $\Gamma(r, R)$ be the family of arcs lying in $A_R^* \setminus A_r^*$ and joining set A_R^* and $M \setminus A_R^*$. It is clear that $\operatorname{mod} \Gamma_R \leq \operatorname{mod} \Gamma(r, R)$. Our goal henceforth is to make a suitable estimate from above of mod $\Gamma(r, R)$.

Take the function (m), which equals $|x(m)|^{-1}$ when $m \in A_R^* \setminus A_r^*$ and vanishes with all other values of $m \in M$. Since

$$|d|x(m)|| \leq \frac{1}{|x(m)|} |(x(m), dx(m))| \leq |dx(m)|,$$

then, for any arc $\gamma \in \Gamma(r, R)$,

$$\int_\gamma \rho(m) \geq \int_\gamma \frac{|d|x(m)||}{|x(m)|} \geq \ln \frac{R}{r}.$$

is satisfied. Thus, the function $\rho_0(m) = \rho(m)(\ln(R/r))^{-1}$ is admissible for the family of arcs $\Gamma(r, R)$ and therefore

$$\operatorname{mod} \Gamma(r, R) \leq \frac{1}{\left(\ln \frac{R}{r}\right)^2} \int_{A_R^* \setminus A_r^*} \frac{1}{|x(m)|^2}. \tag{5}$$

We will estimate the integral on the right side of (5), which expresses the "logarithmic area" of set $A_R \setminus A_r$. It should be noted at the outset that, by the minimality of surface (M, x), the equality

$$\Delta(a, x(m)) = 0, \tag{6}$$

is satisfied for any field a parallel in R^3, where Δ is the Laplace operator in the metric of the surface (see [7], p. 309). Hence it follows that

$$\frac{1}{2} \Delta \ln^2 |x| = 1/|x|^2 + ((x, n)^2/|x|^4)(2 \ln |x| - 1),$$

where n is the unit vector of the normal bundle over (M, x). Applying the Stokes formula to this equality, we obtain

$$\int\limits_{A_R^\circ\setminus A_r^\circ}\frac{1}{|x|^2}=\int\limits_{\partial(A_R^\circ\setminus A_r^\circ)}\frac{\ln|x|}{|x|}\langle\nabla|x|,\nu\rangle-\int\limits_{A_R^\circ\setminus A_r^\circ}\frac{\langle x,n\rangle^2}{|x|^4}(2\ln|x|-1)=$$
$$=\frac{\ln R}{R}\int\limits_{\partial A_R^\circ}\langle\nabla|x|,\nu\rangle-\frac{\ln r}{r}\int\limits_{\partial A_r^\circ}\langle\nabla|x|,\nu\rangle-\int\limits_{A_R^\circ\setminus A_r^\circ}\frac{\langle x,n\rangle^2}{|x|^2}(2\ln|x|-1), \tag{7}$$

where ν is the unit vector of the normal to the boundary $\partial(A_R^\circ\setminus A_r^\circ)$.

There similarly follows from equality (6) $\Delta\ln|x|=(2/|x|^4)\langle x,n\rangle^2$ and, further,

$$\frac{1}{R}\int\limits_{\partial A_R^\circ}\langle\nabla|x|,\nu\rangle-\frac{1}{r}\int\limits_{\partial A_r^\circ}\langle\nabla|x|,\nu\rangle=2\int\limits_{A_R^\circ\setminus A_r^\circ}\frac{\langle x,n\rangle^2}{|x|^4}. \tag{8}$$

Combining (7) and (8), we arrive at the equality

$$\int\limits_{A_R^\circ\setminus A_r^\circ}\frac{1}{|x|^2}=\frac{1}{r}\ln\frac{R}{r}\int\limits_{\partial A_r^\circ}\langle\nabla|x|,\nu\rangle+\int\limits_{A_R^\circ\setminus A_r^\circ}\frac{\langle x,n\rangle^2}{|x|^4}\left(1+2\ln\frac{R}{|x|}\right).$$

Remark that

$$\int\limits_{A_R^\circ\setminus A_r^\circ}\frac{\langle x,n\rangle^2}{|x|^4}\left(1+2\ln\frac{R}{|x|}\right)<\left(1+2\ln\frac{R}{r}\right)\int\limits_{A_R^\circ\setminus A_r^\circ}\frac{|\langle x,n\rangle|}{|x|^3}<\left(1+2\ln\frac{R}{r}\right)\cdot n(R,S),$$

and therefore

$$\int\limits_{A_R^\circ\setminus A_r^\circ}\frac{1}{|x|^2}<\frac{1}{r}\ln\frac{R}{r}\int\limits_{\partial A_r^\circ}|\nabla|x||+\left(1+2\ln\frac{R}{r}\right)\cdot n(R,S).$$

Substituting the resultant estimate into [5] and passing to the limit as R → ∞, we obtain the required assertion.

Now assume surface (M, x) to satisfy condition (1). Let e_1, e_2, e_3 be the standard basis of $R^3, x=\sum_{i=1}^{3}x^ie_i$. It can be assumed with no loss of generality that plane V coincides with the plane $x^3 = 0$. Then $N(x^1, x^2)=N(v)$, where $v=(x^1, x^2, 0)$, is the number of points on x(M) having the necessary first two coordinates. In accordance with condition (1), we will assume that $x^3(M) > 0$ on M. Henceforth put $D(R,\eta)=\{m\in M:|v(m)|<R,\ x^3<\eta\}$.

As previously, we choose r > 0 for a specified compact set $F\subset M$ so that $F\subset D(r,r)$. We take as the exhaustion of manifold M the sequence of sets $D(R_k, R_k)$, where $R_k>r$, $R_k\nearrow\infty$. Fix $R = R_k$ and assume Γ(r, R) to be the family of locally rectifiable arcs joining D(r, r) and $M\setminus D(R,R)$. We will consider that $\rho(m)=|x(m)|^{-1}$ with $m\in D(R,R)\setminus D(r,r)$ and equals zero in other $m\in M$. As in the first case, it is easily shown that the function $\rho_0(m)=\rho(m)\ln(R/r\sqrt{2})$ is admissible for family Γ(r, R) and therefore

$$\mathrm{mod}\,\Gamma(r,\ R) < \frac{1}{\left(\ln \frac{R}{r\sqrt{2}}\right)^2} \int\limits_{D(R,\ R)\setminus D(r,\ r)} \frac{1}{|x|^2}. \tag{9}$$

We will first evaluate the integral $\int\limits_{E(h)} 1/|x(m)|^2$ over the set $E(h) = D(R,\ h)\setminus D(r,\ h)$. Assume $f(m) = (1/|v(m)|)\,\mathrm{arctg}\,(x^3(m)/|v(m)|)$. Using the Stokes formula, we have

$$\int\limits_{E(h)} \langle \nabla x^3,\ \nabla f\rangle = \int\limits_{\partial E(h)} f\,\langle \nabla x^3,\ \nu\rangle, \tag{10}$$

where ν is the unit vector of the normal ∂E(h). One can easily calculate the values of the necessary gradients.

$$\nabla x^3 = e_3^T,\ \nabla v = v^T/|v|,$$

where e^T is the projection of field e on the tangent space at the corresponding point. Consequently,

$$\nabla f = \frac{\partial f}{\partial |v|}\nabla |v| + \frac{\partial f}{\partial x^3}\nabla x^3 = \frac{e_3^T}{|x|^2} - \frac{v^T}{|v|^3}\left(\mathrm{arctg}\frac{x^3}{|v|} + \frac{x^3|v|}{|x|^2}\right)$$

and

$$\frac{|e_3^T|^2}{|x|^2} = \frac{\langle v^T,\ e_3^T\rangle}{|v|^2}\left(\mathrm{arctg}\frac{x^3}{|v|} + \frac{x^3|v|}{|v|^2 + (x^3)^2}\right) + \langle \nabla f,\ \nabla x^3\rangle. \tag{11}$$

By the orthogonality of e_3 and v in R^3, we have

$$\langle v^T,\ e_3^T\rangle = -\langle v,\ n\rangle\langle e_3,\ n\rangle,\ |e_3^T|^2 = 1 - \langle e_3,\ n\rangle^2$$

and from (11) we obtain

$$\frac{1}{|x|^2} = \frac{\langle e_3,\ n\rangle}{|v|^2}\left[\frac{\langle e_3,\ n\rangle}{1+\xi^2} - \frac{\langle v,\ n\rangle}{|v|}\left(\mathrm{arctg}\,\xi + \frac{\xi}{1+\xi^2}\right)\right] + \langle \nabla f,\ \nabla x^3\rangle,$$

where $\xi = x^3/|v|$. Utilizing the Cauchy inequality, we arrive at the estimate

$$\frac{1}{|x|^2} < \frac{|\langle e_3,\ n\rangle|}{|v|^2}\left[\frac{1}{(1+\xi^2)^2} + \left(\mathrm{arctg}\,\xi + \frac{\xi}{1+\xi^2}\right)^2\right]^{1/2} + \langle \nabla f,\ \nabla x^3\rangle <$$
$$< \frac{\pi}{2}\cdot\frac{|\langle e_3,\ n\rangle|}{|v|^2} + \langle \nabla f,\ \nabla x^3\rangle.$$

Substituting this estimate into (10), we have

$$\int\limits_{E(h)} \frac{1}{|x|^2} < \frac{\pi}{2}\int\limits_{E(h)} \frac{|\langle e_3,\ n\rangle|}{|v|^2} + \int\limits_{\partial E(h)} f\,\langle e_3^T,\ \nu\rangle < \frac{\pi}{2}\int\limits_{\pi(E(h))} N(x^1,\ x^2)\frac{dx^1dx^2}{(x^1)^2+(x^2)^2} + \int\limits_{\partial E(h)} f\cdot\langle e_3^T,\ \nu\rangle. \tag{12}$$

Let L(h), γ_t be the segments of boundary ∂E(h) lying on sets $x^3 = h$, $|v| = t$ respectively. The validity of the equality

$$\int\limits_{L(h)} \langle e_3^T,\ \nu\rangle\frac{1}{|v|}\mathrm{arctg}\frac{h}{|v|} = -\int\limits_{\gamma_r \cup \gamma_R} \langle e_3^T,\ \nu\rangle\frac{1}{|v|}\mathrm{arctg}\frac{h}{|v|} + \int\limits_{E(h)} \left\langle \nabla\left(\frac{1}{|v|}\mathrm{arctg}\frac{h}{|v|}\right),\ \nabla x^3\right\rangle. \tag{13}$$

is easily seen. Reasoning in the same manner as above, we can ascertain the correctness of the following estimates:

$$\left|\int\limits_{\partial E(h)} f\cdot\langle e_3^T, \nu\rangle\right| = \left|\int\limits_{L(h)} \frac{\langle e_3^T, \nu\rangle}{|v|}\operatorname{arctg}\frac{h}{|v|} + \int\limits_{\tau_r\cup\tau_R} \frac{\langle e_3^T, \nu\rangle}{|v|}\operatorname{arctg}\frac{x^3}{|v|}\right| =$$

$$= \left|\int\limits_{E(h)} \left\langle \nabla\left(\frac{1}{|v|}\operatorname{arctg}\frac{h}{|v|}\right), \nabla x^3\right\rangle - \int\limits_{\tau_r\cup\tau_R} \frac{\langle e_3^T, \nu\rangle}{|v|}\left(\operatorname{arctg}\frac{h}{|v|} - \operatorname{arctg}\frac{x^3}{|v|}\right)\right| < \tag{14}$$

$$< \frac{\pi}{2}\int\limits_{v_0(E(h))} \frac{N(x^1, x^2)}{|v|^2}\,dx^1dx^2 + \frac{\pi}{2}\int\limits_{\tau_r\cup\tau_R} |\langle e_3^T, \nu\rangle|\frac{1}{|v|}.$$

Further, remark that

$$|\langle e_3^T, \nu\rangle| = |\langle e_3, \nu\rangle| < |e_3^W|,$$

where W is the two-dimensional normal space along $\tau_r\cup\tau_R$, and consequently,

$$\int\limits_{\tau_r\cup\tau_R} |\langle e_3^T, \nu\rangle|\frac{1}{|v|} < \frac{n(R)}{R} + \frac{n(r)}{r},$$

where $n(R) = n(R, V)$. Finally, we obtain from (12), (14)

$$\int\limits_{E(h)} \frac{1}{|x|^2} < \frac{\pi}{2}\left[\frac{n(R)}{R} + \frac{n(r)}{r} + 2\int\limits_r^R \frac{n(t)}{t^2}dt\right]. \tag{15}$$

The final step in the proof entails estimation of the integral $\int\limits_{E(a,b)} 1/|x|^2$, where $E(a, b) = \{m\in M : x^3(m)\in(a, b), |v(m)| < r\}, 0 < a < b$. For this purpose, we first note that

$$\int\limits_{E(a,b)} \frac{1}{|x^3|^2} = \int\limits_{E(a,b)} \left(\frac{\langle e_3, n\rangle^2}{|x^3|^2} - \left\langle \nabla\left(\frac{1}{x^3}\right), \nabla x^3\right\rangle\right) = \int\limits_{E(a,b)} \frac{\langle e_3, n\rangle^2}{|x^3|^2} - \int\limits_{\partial E(a,b)} \frac{1}{x^3}\langle e_3^T, \nu\rangle. \tag{16}$$

The estimation of the first integral

$$\int\limits_{E(a,b)} \frac{\langle e_3, n\rangle^2}{|x^3|^2} < \frac{1}{a^2}\int\limits_{E(a,b)} |\langle e_3, n\rangle| < \frac{1}{a^2}\int\limits_0^r n(t)\,dt,$$

is obvious, as is the equality

$$0 = \int\limits_{\partial E(a,b)} \langle e_3^T, \nu\rangle = \int\limits_{L(b)} \langle e_3^T, \nu\rangle + \int\limits_{L(a)} \langle e_3^T, \nu\rangle + \int\limits_{\tau_r} \langle e_3^T, \nu\rangle.$$

The latter integral is independently bounded away from a, b:

$$\left|\int\limits_{\tau_r} \langle e_3^T, \nu\rangle\right| < n(r),$$

and, consequently, assuming $a = 0$, we have

$$\left|\int\limits_{L(b)}\langle e_3^T, \nu\rangle\right| < n(r).$$

Returning to (16), we have

$$\int\limits_{E(a,b)}\frac{1}{|x^3|^2} < \frac{1}{a^2}\int\limits_0^r n(t)\,dt + \frac{2n(r)}{a} + \frac{n(r)}{b}.$$

Hence we conclude that

$$\int\limits_{E(a,\infty)}\frac{1}{|x|^2} < \int\limits_{E(a,\infty)}\frac{1}{|x^3|^2} < c(r, a) = \frac{1}{a^2}\int\limits_0^r n(t)\,dt + \frac{2n(r)}{a}.$$

Remarking that $D(R, R)\setminus D(r, r) = E(R)\cup E(r, R)$, we have

$$\int\limits_{D(R,R)\setminus D(r,r)}\frac{1}{|x|^2} < \frac{\pi}{2}\left(\frac{n(r)}{r} + \frac{n(R)}{R} + \int\limits_r^R\frac{2n(t)}{t^2}\,dt\right) + c(r),$$

and, taking condition (1) into account, we have the required assertion when $R = R_k \to \infty$.

4. **Proof of theorems 1 and 2.** We can assume with no loss of generality that surface (M, x) lies in the half-space $x^3 > 0$. We conclude from lemmas 1 and 2 that surface (M, x) is parabolic in type. Assume that $x^3(m) \neq \text{const}$ with $m \in M$.

Specify an arbitrary point $m_0 \in M$ and a constant $c > x^3(m_0)$. Denote by O the component of the connection of set $\{m \in M : x^3(m) < c\}$, containing point m_0. It is clear that set O is not empty. Consider the function w(m), which equals $c - x^3(m)$ on set O and vanishes outside O.

Let $F \subset O$ be a compact set, and let $A_1 \subset A_2 \subset \ldots$ be the exhaustion of M by the sequence of open sets $A_k \supset F$ with compact closures, for which property (3) is satisfied. Let $\phi(m)$ be a Lipschitz function admissible in computing the capacity of condenser $(F; M\setminus A_k)$. The function $w_1(m) = w(m)\cdot\varphi^2(m)$ is a Lipschitz function with compact carrier contained in $\bar{O}\cap\bar{A}_k$. Utilizing the Stokes formula and proceeding from (6), we have

$$\int\limits_M\langle\nabla w_1, \nabla x^3\rangle = -\int\limits_M w_1\,\Delta x^3 = 0.$$

Hence it follows that

$$\int\limits_O\varphi^2\cdot|\nabla x^3|^2 = -2\int\limits_O\varphi w\cdot\langle\nabla\varphi, \nabla x^3\rangle.$$

Since $|w(m)| < c$, is fulfilled everywhere on O,

$$\int\limits_O\varphi^2|\nabla x^3|^2 < 2c\left(\int\limits_O\varphi^2|\nabla x^3|^2\right)^{1/2}\cdot\left(\int\limits_O|\nabla\varphi|^2\right)^{1/2}.$$

and therefore

$$\int_O \varphi^2 |\nabla x^3|^2 < 4c^2 \int_O |\nabla \varphi|^2.$$

Since $\phi(m) = 1$ on F, after minimizing the right side of this inequality for all functions $\phi(m)$, we conclude that

$$\int_F |\nabla x^3|^2 < 4c^2 \cdot \operatorname{cap}(F, M \setminus A_k).$$

Taking property (3) of the exhaustion into account, we establish that

$$\int_F |\nabla x^3|^2 = 0.$$

Hence, by the rule of choice for compact $F \subset O$, we arrive at the conclusion that $\nabla x^3 = 0$ everywhere on O. Thus, x^3 = const on O, which contradicts the definition of manifold O. The theorem has been proved.

REFERENCES

1. M. Miranda, "Frontiere minimal con ostacoli," Ann. Univ. Ferrara, Ser. 7, vol. 16, no. 2, pp. 29-37, 1971.
2. V. M. Miklyukov, "Some properties of tubular minimal surfaces in R^n," DAN SSSR, vol. 247, no. 3, pp. 549-552, 1979.
3. L. P. de M Jorge and F. A. Xavier, "A complete minimal surface in R^3 between two parallel planes," Ann. Math., vol. 112, no. 1, pp. 203-206, 1980.
4. S. Y. Cheng and S. T. Yau, "Differential equations on Riemannian manifold and their geometric applications," Communs. Pure Appl. Math., vol. 28, no. 3, pp. 333-354, 1975.
5. V. M. Miklyukov, "A new approach to the Bernshtein theorem and related problems of equations of minimal surface type," Matem. sb., vol. 108(150):2, pp. 268-289, 1979.
6. B. Fuglede, "Extremal length and functional complection," Acta Math., vol.98, no. 3-4, pp. 171-219, 1957.
7. S. Kobayashi and K. Nomizu, Principles of Differential Geometry [Russian translation], vol. 2, Moscow, 1981.

23 May 1984 Volgograd