A Jordan algebra approach to the eiconal equation

Vladimir Tkachev[*]

November 30, 2012

Abstract

We establish a natural correspondence between (the equivalence classes of) cubic solutions of $\sum_{i,j=1}^{k} Q^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} = 9(\sum_{i,j=1}^{k} Q_{ij} x_i x_j)^2$ and (the isomorphy classes of) cubic Jordan algebras.

1 Introduction

The Jordan-Von Neumann-Wigner classification [9] implies that the only possible formally real rank 3 Jordan algebras J are:

	a Jordan algebra, J	the generic norm of a trace free element, $\sqrt{2}N(x)$
(i)	$\mathrm{Herm}_3(\mathbb{F}_d),\ d = 1, 2, 4, 8$	$u_d(x)$
(ii)	$\mathbb{R} \oplus \mathrm{Spin}(\mathbb{R}^{n+1})$	$4x_n^3 - 3x_n \lvert x \rvert^2$
(iii)	$\mathbb{F}_1^3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	$x_2^3 - 3x_2 x_1^2$

where J is equipped with the standard inner product $\langle x, y \rangle = \frac{1}{3} \mathrm{Tr}(x \bullet y)$ and

$$u_d(x) = x_{3d+2}^3 + \frac{3}{2} x_{3d+2}(\lvert z_1 \rvert^2 + \lvert z_2 \rvert^2 - 2\lvert z_3 \rvert^2 - 2x_{3d+1}^2)$$
$$+ \frac{3\sqrt{3}}{2} x_{3d+1}(\lvert z_2 \rvert^2 - \lvert z_1 \rvert^2) + 3\sqrt{3}\,\mathrm{Re}(z_1 z_2 z_3),$$

is a Cartan isoparametric cubic. Here $z_k = (x_{kd-d+1}, \ldots, x_{kd}) \in \mathbb{R}^d = \mathbb{F}_d$ and $\mathbb{F}_d$ stands for a real division algebra of dimension d, i.e. $\mathbb{F}_1 = \mathbb{R}$ (the reals), $\mathbb{F}_2 = \mathbb{C}$ (the complexes), $\mathbb{F}_4 = \mathbb{H}$ (the quaternions) and $\mathbb{F}_8 = \mathbb{O}$ (the octonions). According to a recent result of the author [16], the cubic forms displayed in the last column are exactly the only possible cubic solutions of

$$\lvert \nabla u(x) \rvert^2 = 9\lvert x \rvert^4, \quad x \in \mathbb{R}^n. \tag{1.1}$$

This yields a natural question: Does there exist any *explicit* correspondence between cubic solutions of (1.1) and formally real cubic Jordan algebras?

The concept of Jordan algebra, and formally real Jordan algebra especially, was introduced originally in the 1930s in [8], [9] in attempt to find an adequate algebraic formalism for quantum mechanics. Nowadays, besides the numerous pure algebraic applications (in particular, the connection between Jordan and Lie theory via the Tits-Kantor-Koecher construction of Lie algebras), Jordan algebras appeared to be very useful in harmonic analysis on self-dual homogeneous cones (Vinberg-Koecher theory), operator theory (JB-algebras), differential geometry (symmetric spaces, projective geometry, isoparametric hypersurfaces), integrable hierarchies (Svinopulov's characterization of generalized KdV equation), statistics (Wishart distributions on Hermitian matrices and on Euclidean Jordan algebras).

[*]Department of Mathematics, Linköping University, SE-581 83, Linköping, Sweden, tkatchev@kth.se

In the most interesting for us case of cubic Jordan algebras, the multiplication structure can be recovered directly from the generic norm by the so-called Springer construction. The cubic case is also very distinguished because its very remarkable appearance in different areas such as Scorza and secant varieties, Cremona transformations, non-classical solutions to fully nonlinear PDEs, extremal black holes and supergravity [2], [3], [1], [4], [10], [12], [13], [14].

In this paper we obtain an equivalent description of cubic Jordan algebras in terms of cubic solutions of the following eiconal type equation:

$$\sum_{i,j=1}^{k} Q^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} = 9 \Big(\sum_{i,j=1}^{k} Q_{ij} x_i x_j \Big)^2, \quad x \in \mathbb{R}^k, \tag{1.2}$$

where Q_{ij} is an invertible matrix with Q^{ij} being its inverse. It is more natural, however, to work with an invariant version of (1.2) which we now define. Let (W, Q) be a nondegenerate quadratic space, i.e. W be a vector space over a field $\mathbb{F}$ ($\mathbb{F}$ is always assumed to be $\mathbb{R}$ or $\mathbb{C}$) with a quadratic form $Q : W \to \mathbb{F}$ such that the associated symmetric bilinear form

$$Q(x; y) = \frac{1}{2}(Q(x+y) - Q(x) - Q(y)), \qquad x, y \in W$$

has the kernel $\{0\}$. The bilinear form $Q(x; y)$ induces a natural inner product on W. Let $\nabla u(x)$ denote the covariant derivative of a smooth function $u : W \to \mathbb{F}$, i.e. the unique vector field satisfying the dual realtion

$$Q(\nabla u(x); y) = \partial_y u|_x, \qquad \forall y \in W. \tag{1.3}$$

Definition 1.1. A triple (W, Q, u) with a cubic form $u : W \to \mathbb{F}$ on a nondegenerate quadratic space (W, Q) satisfying

$$Q(\nabla u(x)) = 9 Q(x)^2, \quad x \in W \tag{1.4}$$

is called a *eiconal triple*.

Remark 1.1. It is easy to check that in the local coordinates $x_i = Q(x, e_i)$ associated with a basis $\{e_i\}_{1 \le i \le n}$ of W, the eiconal equation (1.4) takes the form (1.2), where $Q_{ij} = Q(e_i; e_j)$. In particular, (1.4) is obtained for the particular choice $W = \mathbb{R}^n$ and $Q(x) = |x|^2 \equiv \sum_{i=1}^{n} x_i^2$.

Recall that a map $O : (W, Q) \to (\tilde{W}, \tilde{Q})$ between the quadratic spaces (W, Q) and $(\tilde{W}, \tilde{Q})$ is called an isometry if O is invertible and $\tilde{Q}(Ox) = Q(x)$ for all $x \in W$. It is straightforward to verify that if (W, Q, u) is an eiconal triple and $O : (W, Q) \to (\tilde{W}, \tilde{Q})$ is an isometry then $(\tilde{W}, \tilde{Q}, \tilde{u})$ with $\tilde{u}(Ox) = u(x)$ is an eiconal triple too. We call such eiconal triples *equivalent*.

Theorem 1. *There exists a correspondence* $\mathcal{J} \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{J}$, $\alpha\beta = \mathrm{id}_{\mathcal{E}}$, $\beta\alpha = \mathrm{id}_{\mathcal{J}}$, *between the equivalence classes of eiconal triples ($\mathcal{E}$) and the isomorphic classes of cubic Jordan algebras ($\mathcal{J}$) and defined as follows.*

(α) *Any cubic Jordan algebra* $J = J(V, N, e)$ *with the generic norm N gives rise to an eiconal triple* $(e^{\perp}, \tau, u)$ *with the underlying vector space* $e^{\perp} := \{x : \mathrm{Tr}\, x = 0\}$, *the quadratic form* $\tau(x) = \frac{1}{3}\mathrm{Tr}(x^2)|_{e^{\perp}}$, *and* $u(x) = \sqrt{2}\, N(x)$. *Two isomorphic algebras give rise to equivalent eiconals.*

(β) *In the converse direction, any eiconal triple* (W, Q, u) *produces a cubic Jordan algebra* $J = J(V, N, e)$ *with the underlying vector space* $V = \mathbb{F} \times W$, *the generic norm* $N(\mathbf{x}) = x_0^3 - \frac{3}{2}x_0 Q(x) + \frac{u(x)}{\sqrt{2}}$, *and the unit* $e = (1, 0)$, *where* $\mathbf{x} = (x_0, x) \in V$. *Two equivalent eiconals produces isomorphic algebras.*

We give the basic information on Jordan algebras in section 2 and then prove Theorem 1 in sections 3 and 4.

2 Basic facts on Jordan algebras

Here we review some definitions in Jordan algebra theory following the classical books [7], [11] and [5]. A vector space[1] V over a field $\mathbb{F}$ with a commutative $\mathbb{F}$-bilinear map $\bullet : V \times V \to V$ is called a Jordan algebra if it satisfies

$$x^2 \bullet (x \bullet y) = x \bullet (x^2 \bullet y), \qquad (2.1)$$

where $x^2 := x \bullet x$. Any Jordan algebra is power associative, i.e. the subalgebra $V(x)$ generated by a single element x is associative for any $x \in V$. The rank of V is $r := \max\{\dim V(x) : x \in V\}$. An element x is said to be *regular* if $\dim V(x) = r$. For a regular $x \in V$, the elements $e, x, \dots, x^{r-1}$ are linearly independent, so that

$$x^r = \sigma_1(x)x^{r-1} - \sigma_2(x)x^{r-2} + \dots + (-1)^{r-1}\sigma_r(x)e,$$

where $\sigma_i(x) \in \mathbb{F}$ are uniquely defined by x. The polynomial

$$m_x(\lambda) = \lambda^r - \sigma_1(x)\lambda^{r-1} + \dots + (-1)^r \sigma_r(x) \in \mathbb{F}[\lambda]$$

is called the *minimum polynomial* of the regular element $x \in V$ [6]. The coefficients $\sigma_i(x)$ are polynomial functions of x homogeneous of degree i in the following sense. If $\{e_i\}_{1 \le i \le n}$ is an arbitrary basis of V and

$$x_\xi := \sum_{i=1}^{n} \xi_i e_i, \qquad \xi \in \mathbb{F}^n,$$

is a generic element of the algebra V then

$$x_\xi^r - \sigma_1(\xi)x_\xi^{r-1} + \dots + (-1)^r \sigma_r(\xi)e = 0,$$

where $\sigma(\xi)$ is a polynomial function of x homogeneous of degree i.

Definition. The coefficient $\sigma_1(x) = \mathrm{Tr}\, x$ is called the *generic trace* of x, and $\sigma_n(x) = N(x)$ is called the *generic norm* of x.

Example 2.1. A standard example of a Jordan algebra is the vector space $M_n(\mathbb{F}_d)$ of matrices of size $n \times n$ with entries in $\mathbb{F}_1 = \mathbb{R}$ or $\mathbb{F}_2 = \mathbb{C}$, and the bullet product

$$x \bullet y := \frac{1}{2}(xy + yx). \qquad (2.2)$$

In this case, $\mathrm{Tr}\, x = \mathrm{tr}\, x$, $N(x) = \det x$ are the usual trace and determinant of x. In fact, it is readily verified that the real vector space $\mathrm{Herm}_3(\mathbb{F}_d)$ of self-adjoint matrices over the real division algebra $\mathbb{F}_d$ with $d = 1, 2, 4$ and arbitrary $n \ge 1$, and with $d = 8$ and $n \le 3$ is also a Jordan algebra with respect to the multiplication (2.2).

Example 2.2. The following well-known construction associates a Jordan algebra structure to an arbitrary quadratic space (V, Q) over $\mathbb{F}$ with a basepoint e, i.e. $Q(e) = 1$, by virtue of the following multiplication

$$x \bullet y = Q(x; e)y + Q(y; e)x - Q(x; y)e. \qquad (2.3)$$

It is readily that (2.3) satisfies the Jordan identity (2.1) with the unit element e. Thus obtained Jordan algebra $\mathrm{Spin}(V, Q, e)$ is called a *spin factor* (or a quadratic factor). It follows from (2.3) that any element $x \in V$ satisfies the quadratic identity

$$x^2 - 2Q(x, e)x + Q(x)e = 0$$

which, in particular, shows that $\mathrm{Tr}(x) = 2Q(x, e)$ and $N(x) = Q(x)$. Conversely, any rank 2 Jordan algebra is obtained by virtue of the above construction from an appropriate quadratic space. In any case, the multiplicative structure on a quadratic Jordan algebra is completely determined by its generic norm.

---

[1]We always assume that V is finite dimensional.

In the case of Jordan algebra of rank 3 one still has a possibility to restore the multiplicative structure from the generic norm, but in contrast to the quadratic case only very special cubic forms can be used to construct cubic Jordan algebras. The corresponding procedure is called the Springer construction [15] and we recall below following [11, II.4.2]. Let V be a vector space over a field $\mathbb{F}$ with a cubic form $u : V \to V$. Notice that the set $u^{-1}(1) = \{x \in V : u(e) = 1\}$ is nonempty because u is an odd function. A vector $e \in u^{-1}(1)$ is called a basepoint. Consider the decomposition

$$u(x + ty) = u(x) + tu(x; y) + t^2 u(y; x) + t^3 u(y), \qquad x, y \in V, \ t \in \mathbb{F}, \tag{2.4}$$

where $u(x; y) = \partial_y u|_x$ is the directional derivative of u in the direction y evaluated at x. Since $u(x; y)$ is a quadratic form with respect to the first variable, one can polarize it to obtain the full linearization of the cubic form u:

$$u(x; y; z) = u(x + y; z) - u(x; z) - u(y; z) = \partial_x \partial_y \partial_z u|_e. \tag{2.5}$$

Then Euler's homogeneity function theorem implies

$$u(x) = \frac{1}{3} u(x; x) = \frac{1}{6} u(x; x; x). \tag{2.6}$$

Now we fix some cubic form $N(x) \not\equiv 0$ on V and its basepoint $e \in N^{-1}(1)$. Linearizing N as above, one defines the *linear trace* form and and the *quadratic spur* form to be

$$\mathrm{Tr}(x) = N(e; x) = N(e; e; x) \tag{2.7}$$

and

$$S(x) = N(x; e) = N(x; x; e), \tag{2.8}$$

respectively. Then

$$\mathrm{Tr}(e) = S(e) = 3N(e) = 3. \tag{2.9}$$

Polarizing the quadratic spur form,

$$S(x; y) = N(x; y; e) \equiv S(x + y) - S(x) - S(y)$$

we define the bilinear *trace form* by

$$T(x; y) = \mathrm{Tr}(x)\,\mathrm{Tr}(y) - S(x; y) \equiv -\partial_x \partial_y \log N_{ce} \tag{2.10}$$

Definition 2.1. Let V be a finite-dimensional vector space over $\mathbb{F}$ and let $N : V \to \mathbb{R}$ be a cubic form with basepoint e. N is said to be a *Jordan cubic form*, if the following is satisfied:

(i) the trace bilinear form $T(x; y)$ is a nondegenerate bilinear form;

(ii) the quadratic sharp map $\# : V \to V$, defined (uniquely) by

$$T(x^{\#}; y) = N(x; y), \tag{2.11}$$

satisfies the adjoint identity

$$(x^{\#})^{\#} = N(x)x. \tag{2.12}$$

Example 2.3. We demonstrate the above definitions by the Jordan algebra of symmetric 3×3 matrices $V = \mathrm{Herm}_3(\mathbb{R})$ with the unit matrix being a basepoint: $e = \mathbf{1}$ and a cubic form $N(x) = \det x$. Then

$$\mathrm{Tr}\, x = \partial_x N|_e = (\det(\mathbf{1} + tx))'_{t=0} = \mathrm{tr}\, x,$$

and, by virtue of the Newton identities,

$$S(x) = \partial_e N|_x = (\det(x + t\mathbf{1}))'_{t=0} = \frac{1}{2}((\mathrm{tr}\, x)^2 - \mathrm{tr}\, x^2).$$

4

Polarizing the latter identity yields $S(x; y) = \operatorname{tr} x \operatorname{tr} y - \operatorname{tr} x \bullet y$. Thus,

$$T(x; y) = \operatorname{tr} x \bullet y \equiv \operatorname{tr} xy.$$

Now observe that $T(x; y) = \operatorname{tr} xy$ obviously is a nondegenerate bilinear form, hence we get (i) in Definition 2.1. On the other hand, if x is an invertible matrix then

$$3N(x; y) = \partial N_y|_x = (\det(x + ty))'_{t=0} = (\det x \, \det(\mathbf{1} + tyx^{-1}))'_{t=0} = \det x \, \operatorname{tr}(yx^{-1}),$$

which yields by (2.11) the adjoint map $x^{\#} = x^{-1} \det x$ is the adjoint matrix in the usual sense of linear algebra, thus $x^{\#\#} = x \det x$ and the adjoint identity (2.12) follows.

Proposition 2.1 (The Springer construction, [11], p. 77). *To any Jordan cubic form $N : V \to \mathbb{F}$ with basepoint $e \in V$ one can associate a unital Jordan algebra $\mathcal{J}(V, N, e)$ of rank 3 with the unit e and the Jordan multiplication determined by the formula*

$$x \bullet y = \frac{1}{2}(x \# y + \operatorname{Tr}(x)y + \operatorname{Tr}(y)x - S(x; y)e), \tag{2.13}$$

where $x \# y = (x + y)^{\#} - x^{\#} - y^{\#}$. All elements of $\mathcal{J}(V, N, e)$ satisfy the cubic identity

$$x^3 - \operatorname{Tr}(x)x^2 + S(x)x - N(x)e = 0, \tag{2.14}$$

and the sharp identity

$$x^{\#} = x^2 - \operatorname{Tr}(x)x + S(x)e. \tag{2.15}$$

We mention some useful consequences of the above construction. By (2.11) $T(x^{\#}; e) = N(x; e) = S(x)$. Polarizing this identity one obtains $T(x \# y; e) = S(x; y)$. On the other hand, (2.10) and (2.9) yield

$$T(x; e) = \operatorname{Tr}(x) - S(x; e) = \operatorname{Tr}(x) - N(x; e; e) = \operatorname{Tr}(x),$$

hence

$$\operatorname{Tr}(x \# y) = T(x \# y; e) = S(x; y). \tag{2.16}$$

Applying the trace to (2.15) and using the last identity one readily finds that

$$S(x; y) = \operatorname{Tr}(x) \operatorname{Tr}(y) - \operatorname{Tr}(x \bullet y), \tag{2.17}$$

which in its turn yields

$$\operatorname{Tr}(x \bullet y) = T(x; y). \tag{2.18}$$

Setting $x = y$ in (2.16) yields

$$\operatorname{Tr}(x^{\#}) = S(x). \tag{2.19}$$

3 An auxiliary lemma

Here we establish some auxiliary formulas for the generic norm which also have an independent interest. Let $N : V \to \mathbb{F}$ be a Jordan cubic form with basepoint e and let $J = J(V, N, e)$ be the associated cubic Jordan algebra. Notice that the bilinear trace form $T(x; y)$, and thus the inner product

$$\tau(x; y) := \frac{1}{3}T(x; y) \tag{3.1}$$

and the associated quadratic norm $\tau(x) = \tau(x, x)$ are all nondegenerate on V. The normalized factor is chosen to ensure that the basepoint e becomes a unit vector in the induced inner structure. Let us denote by D the covariant derivative on (V, T). Then $\partial_y N|_x = \tau(DN(x); y)$ for any $y \in V$, hence the sharp map definition (2.11) yields

$$\tau(x^{\#}; y) = \frac{1}{3}N(x; y) = \frac{1}{3}\partial_y N|_x = \frac{1}{3}\tau(DN(x); y),$$

therefore by the nondegenerateity of T

$$DN(x) = 3x^{\#}. \tag{3.2}$$

Let us also denote

$$t := \tau(x; e) = \frac{1}{3}\operatorname{Tr}(x). \tag{3.3}$$

It is natural to think of t as a 'time' coordinate function on V. Then the orthogonal complement to the time-line $\mathbb{F}e$ coincides with the subspace of trace free elements of J,

$$e^{\perp} := \{x \in J : \tau(x; e) = 0\} = \{x \in J : \operatorname{Tr}(x) = 0\},$$

which can also be thought as the spatial subspace. This yields the orthogonal decomposition $V = \mathbb{F}e \oplus e^{\perp}$ and the corresponding decomposition of the covariant derivative into the time- and the spatial components,

$$D = \partial_t \oplus \nabla, \tag{3.4}$$

where ∇ denotes the covariant derivative induced by D on $e^{\perp}$.

Lemma 3.1. *In the made notation,*

$$\partial_t N(x) = S(x), \tag{3.5}$$

$$\tau(DN(x)) = 3(\partial_t N(x))^2 - 18tN(x), \tag{3.6}$$

$$\tau(DN(x); D\tau(DN(x)) = 108\,\tau(x)N(x). \tag{3.7}$$

Proof. The first relation follows from the definition of t and (2.19):

$$\partial_t N(x) \equiv \tau(DN(x); e) = \operatorname{Tr}(x^{\#}) = S(x). \tag{3.8}$$

Next, applying (2.10), (2.19) and (2.12), we find

$$\begin{aligned}
\tau(DN(x)) &\equiv 3T(x^{\#}, x^{\#}) \\
&= 3\operatorname{Tr}(x^{\#})^2 - 3S(x^{\#}; x^{\#}) \\
&= 3\operatorname{Tr}(x^{\#})^2 - 6S(x^{\#}) \\
&= 3S(x)^2 - 6\operatorname{Tr}(x^{\#\#}) \\
&= 3S(x)^2 - 18N(x)t,
\end{aligned} \tag{3.9}$$

hence (3.5) yields

$$\tau(DN(x)) = 3(\partial_t N(x))^2 - 18N(x)t, \tag{3.10}$$

thus (3.6) follows.

Finally, in order to show (3.7), we rewrite (3.10) by virtue (3.5) and (3.3) as

$$\tau(DN(x)) = 3S(x)^2 - 6N(x)\operatorname{Tr}(x),$$

which yields by virtue of (2.19)

$$\begin{aligned}
\frac{1}{3}\tau(DN(x); D\tau(\nabla N(x)) &\equiv \partial_{x^{\#}}\tau(\nabla N(x)) \\
&= 6S(x; x^{\#})S(x) - 6N(x)\operatorname{Tr}(x^{\#}) - 6N(x; x^{\#})\operatorname{Tr}(x) \\
&= 6(S(x; x^{\#}) - N(x))\operatorname{Tr}x^{\#} - 6T(x^{\#}; x^{\#})\operatorname{Tr}(x)
\end{aligned} \tag{3.11}$$

By (2.15), $x \bullet x^{\#} = N(x)e$, hence using (2.17), we find

$$S(x; x^{\#}) = \operatorname{Tr}(x)\operatorname{Tr}(x^{\#}) - \operatorname{Tr}(x \bullet x^{\#}) = \operatorname{Tr}(x)\operatorname{Tr}(x^{\#}) - 3N(x).$$

6

On the other hand, by (3.9)

$$T(x^\#; x^\#) = S(x)^2 - 2N(x)\operatorname{Tr}(x) = (\operatorname{Tr}(x^\#))^2 - 2N(x)\operatorname{Tr}(x),$$

and substituting the found identities into (3.11) and using (2.10) yields the required relation:

$$\begin{aligned}
\frac{1}{3}\tau(\,DN(x);\, D\tau(\nabla N(x)) &= 12\,(\operatorname{Tr}(x)^2 - 2\operatorname{Tr}(x^\#))N(x) \\
&= 12\,(\operatorname{Tr}(x)^2 - S(x,x))N(x) \\
&= 12\,T(x,x)N(x) \equiv 36\,\tau(x)N(x).
\end{aligned} \tag{3.12}$$

$\square$

Remark 3.1. We demonstrate the above equations for a formally real Jordan algebra J. Observe that in this case J is Euclidean, i.e. the bilinear form τ is positive definite, see for instance [5, p. 46, p. 154], hence one can choose local coordinates in V such that $\tau(x) = |x|^2$ and $t = x_n$. Since $2S(x) = (\operatorname{Tr} x)^2 - \operatorname{Tr} x^2 = 9x_n^2 - 3|x|^2$, the equation (3.5) and (3.6) yield respectively

$$\partial_{x_n} N(x) = \frac{3}{2}(3x_n^2 - |x|^2),$$
$$|\,DN(x)|^2 = 3(\partial_{x_n} N(x))^2 - 18x_n N(x),$$

while (3.7) shows that $N(x)$ satisfies an eigenfunction type equation

$$\Delta_\infty N(x) = 108|x|^2 N(x)$$

for the ∞-Laplace operator $\Delta_\infty u = \frac{1}{2}\nabla u \cdot \nabla|\nabla u|^2$.

4 The proof of Theorem 1

The proof of the theorem follows from Propositions 4.1, Proposition 4.2 and Proposition 4.3 below.

Proposition 4.1. *Let $J = J(V, N, e)$ be a cubic Jordan algebra, $e^\perp = \{x \in V : \operatorname{Tr} x = 0\}$, $u(x) = \sqrt{2}N(x)|_{e^\perp}$ and the inner product τ is defined by (3.1). Then $(e^\perp, \tau, u)$ is an eiconal triple. Moreover, isomorphic Jordan algebras give rise to equivalent eiconal triples.*

Proof. Notice that by (3.4), $\tau(\,DN(x)) = \tau(\nabla N(x)) + (\partial_t N(x))^2$, hence (3.6) yields

$$\tau(\nabla N(x)) = 2(\partial_t N(x))^2|_{t=0}, \qquad x \in e^\perp. \tag{4.1}$$

On the other hand, by (2.10), $S(x) = -\frac{1}{2}T(x,x) = -\frac{3}{2}\tau(x)$ for any $x \in e^\perp$, thus $\partial_t N(x)|_{t=0} = -\frac{3}{2}\tau(x)$ by (3.5) and setting the found relation into (4.1) yields

$$\tau(\nabla N(x)) = \frac{9}{2}\tau(x)^2 \tag{4.2}$$

which implies that $(e^\perp, \tau, \sqrt{2}N|_{e^\perp})$ is an eiconal triple, as required.

Now we suppose that J_1 and J_2 are isomorphic Jordan algebras and denote by $\phi : J_1 \to J_2$ the corresponding isomorphism. Observe that any element $\xi \in J_2$ satisfies the characteristic relation

$$\xi^3 - \operatorname{Tr}_2(\xi)\xi^2 + S_2(\xi)\xi - N_2(\xi)e_2 = 0,$$

where Tr_k, S_k, N_k and e_k denote the generic trace, the generic spur, the generic norm and the unit element in the algebra J_k, $k = 1, 2$. Applying ϕ^{-1} to the latter relation we find by isomorphy of ϕ that

$$(\phi^{-1}(\xi))^3 - \operatorname{Tr}_2(\xi)(\phi^{-1}(\xi))^2 + S_2(\xi)\phi^{-1}(\xi) - N_2(\xi)e_1 = 0.$$

Setting $x = \phi^{-1}(\xi)$ in the latter relation, we find that any element $x \in J_1$ satisfies the cubic relation

$$x^3 - \mathrm{Tr}_2(\phi(x))x^2 + S_2(\phi(x))x - N_2(\phi(x))e_1 = 0.$$

Since the minimum polynomial is unique, see, for instance [6], the latter relation implies $\mathrm{Tr}_1(x) = \mathrm{Tr}_2(\phi(x))$, $S_1(x) = S_2(\phi(x))$ and

$$N_1(x) = N_2(\phi(x)) \tag{4.3}$$

for any $x \in J_1$. Using (2.10) one finds $T_1(x; y) = T_1(\phi(x); \phi(y))$, which yields $\tau_1(x; y) = \tau_1(\phi(x); \phi(y))$. Since $\phi(e_1) = e_2$ and ϕ is an isomorphism on the level of vector space, one has $\phi(e_1^\perp) = e_2^\perp$, hence $\phi : e_1^\perp \to e_2^\perp$ is an isometry of the quadratic vector spaces $(e_1^\perp, \tau_1)$ and $(e_2^\perp, \tau_2)$, and, moreover, (4.3) implies

$$N_1|_{e_1^\perp}(x) = N_2|_{e_2^\perp}(\phi(x)),$$

thus $(e_1^\perp, \tau_1, N_1|_{e_1^\perp})$ and $(e_2^\perp, \tau_2, N_2|_{e_2^\perp})$ are equivalent, which finishes the proof.

$\square$

Proposition 4.2. *Let (W, Q, u) be an eiconal triple. Then the cubic form*

$$N(\mathbf{x}) \equiv N_u(\mathbf{x}) := x_0^3 - \frac{3x_0 Q(x)}{2} + \frac{u(x)}{\sqrt{2}}, \quad \mathbf{x} = (x_0, x) \in V, \tag{4.4}$$

on the vector space $V = \mathbb{F} \times W$ is Jordan with respect to the basepoint $\mathbf{e} = (1, 0)$. In particular, the associated trace bilinear form is

$$T(\mathbf{x}; \mathbf{y}) = 3x_0 y_0 + 3Q(x; y) \tag{4.5}$$

and the Jordan multiplication structure on $J(V, N, c)$ is determined explicitly by

$$\mathbf{x}^2 = \left(x_0^2 + Q(x),\, 2x_0 x + \frac{1}{3\sqrt{2}} \nabla u(x)\right), \quad \mathbf{x} = (x_0, x). \tag{4.6}$$

Two equivalent eiconal triples produce isomorphic Jordan algebras.

Proof. Our first step is to establish an auxiliary relation (4.16) below. To this end, we recall that

$$Q(\nabla u(x); y) = \partial_y u|_x \equiv u(x; y) \quad \forall y \in W, \tag{4.7}$$

and notice that the covariant gradient of u is a quadratic endomorphism of W and its normalized polarization

$$h(x; y) = \frac{1}{2}(\nabla u(x + y) - \nabla u(x) - \nabla u(y)) \tag{4.8}$$

is obviously a symmetric $\mathbb{F}$-bilinear map. It follows from (4.8) and the homogeneity of $\nabla u(x)$ that

$$h(x) := h(x; x) = \nabla u(x). \tag{4.9}$$

Combining (4.8) and (4.7) with (2.5) we find

$$2Q(h(x; y); z) = Q(\nabla u(x + y) - \nabla u(x) - \nabla u(y); z) = u(x + y; z) - u(x; z) - u(y; z) = u(x; y; z)$$

for all $x, y, z \in V$ which implies

$$Q(x; h(y; z)) = Q(h(x; y)\, z). \tag{4.10}$$

Now rewrite (1.4) by virtue of (4.9) as

$$Q(h(x)) = 9Q^2(x), \tag{4.11}$$

hence applying the directional derivative in the direction y to the latter equation yields

$$Q(h(x); h(x; y)) = 9Q(x)\, Q(y; x) \equiv 9Q(y; Q(x)x), \tag{4.12}$$

8

where $Q(x, y)$ is defined by (1). Using (4.10) we find for the left hand side of (4.12) that $Q(h(x); h(x; y))) = Q(y; h(x; h(x)))$, therefore by the non-degeneracy of Q,

$$h(x; h(x)) = 9Q(x)x. \tag{4.13}$$

Polarizing the latter identity, we obtain

$$h(y; h(x)) + 2h(x; h(x; y)) = 18Q(x; y)x + 9Q(x)y,$$

thus, setting $y = h(x)$ in the obtained identity yields

$$h(h(x)) + 2h(x; h(x; h(x))) = 18Q(x; h(x))x + 9Q(x)h(x), \tag{4.14}$$

where we have used $h(h(x); h(x)) = h(h(x))$. Moreover, by (4.13)

$$h(x; h(x; h(x))) = 9Q(x)h(x; x) = 9Q(x)h(x),$$

and by (4.9) and the homogeneity of u

$$Q(x; h(x)) = Q(x; \nabla u(x)) = \partial_x u|_x = 3u(x), \tag{4.15}$$

therefore (4.14) becomes

$$h(h(x)) = 18Q(x; h(x))x - 9Q(x)h(x) = 54u(x)x - 9Q(x)h(x). \tag{4.16}$$

Now we are ready to verify that (4.4) is Jordan. Notice that $N(\mathbf{e}) = 1$, hence $\mathbf{e}$ is a basepoint of N. Let $\mathbf{x} = (x_0, x)$ and $\mathbf{y} = (y_0, y)$ are chosen arbitrarily in $V = \mathbb{F} \times W$. Since by (4.10)

$$Q(x; h(x; y)) = Q(y; h(x; x)) = Q(y; h(x)),$$

we find

$$N(\mathbf{x}; \mathbf{y}) \equiv \partial_{\mathbf{y}} N|_{\mathbf{x}} = 3\left(x_0^2 - \frac{Q(x)}{2}\right)y_0 - 3x_0 Q(x; y) + \frac{Q(y; h(x))}{\sqrt{2}}. \tag{4.17}$$

Setting $\mathbf{x} = \mathbf{e}$ in (4.17) we find by (2.7) that $\mathrm{Tr}(\mathbf{y}) = \partial_{\mathbf{y}} N|_{\mathbf{e}} = 3y_0$, and setting $\mathbf{y} = \mathbf{e}$ in (4.17), we find by (2.8) that $S(\mathbf{x}) = \partial_{\mathbf{e}} N|_{\mathbf{x}} = 3(x_0^2 - \frac{Q(x)}{2})$, hence

$$S(\mathbf{x}; \mathbf{y}) = S(\mathbf{x} + \mathbf{y}) - S(\mathbf{x}) - S(\mathbf{y}) = 6x_0 y_0 - 3Q(x; y),$$

and therefore (2.10) yields

$$T(\mathbf{x}; \mathbf{y}) = \mathrm{Tr}(\mathbf{x}) \mathrm{Tr}(\mathbf{y}) - S(\mathbf{x}; \mathbf{y}) = 3x_0 y_0 + 3Q(x; y), \tag{4.18}$$

thus proving (4.5). In order to establish the adjoint identity (2.12) note that Q is nondegenerate by the assumption, hence T is nondegenerate too, which implies that $\mathbf{x}^\#$ is well-defined. To determine the adjoint element explicitly we note that $N(\mathbf{x}; \mathbf{y}) = T(\mathbf{x}^\#; \mathbf{y})$, hence comparing (4.17) and (4.18) readily yields

$$\mathbf{x}^\# \equiv (x_0^\#, x^\#) = \left(x_0^2 - \frac{Q(x)}{2}, \frac{h(x)}{3\sqrt{2}} - x_0 x\right) \tag{4.19}$$

Let us denote $\mathbf{x}^{\#\#} \equiv (x_0^{\#\#}, x^{\#\#})$. Then by (4.19)

$$x_0^{\#\#} = (x_0^\#)^2 - \frac{Q(x^\#)}{2} = \left(x_0^2 - \frac{Q(x)}{2}\right)^2 - \frac{Q(x^\#)}{2} \tag{4.20}$$

where by virtue (4.15) and (4.11),

$$Q(x^\#) = Q\left(\frac{h(x)}{3\sqrt{2}} - x_0 x\right)$$

$$= x_0^2 Q(x) - \frac{\sqrt{2}}{3} x_0 Q(x; h(x)) + \frac{Q(h(x))}{18}$$

$$= x_0^2 Q(x) - \sqrt{2} x_0 u(x) + \frac{Q(x)^2}{2},$$

and substituting this into (4.20) yields

$$x_0^{\#\#} = N(\mathbf{x})x_0. \tag{4.21}$$

Arguing as above one obtains

$$
\begin{aligned}
x^{\#\#} &= -x_0^{\#}x^{\#} + \frac{1}{3\sqrt{2}} h(x^{\#}) \\
&= (x_0^2 - \frac{Q(x)}{2})(x_0 x - \frac{h(x)}{3\sqrt{2}}) + \frac{1}{3\sqrt{2}} h\left(\frac{h(x)}{3\sqrt{2}} - x_0 x\right),
\end{aligned}
\tag{4.22}
$$

where using (4.13), (4.16) one has

$$
\begin{aligned}
h\left(\frac{h(x)}{3\sqrt{2}} - x_0 x\right) &= \frac{h(h(x))}{18} - \frac{\sqrt{2}\,x_0}{3} h(x; h(x)) + x_0^2 h(x) \\
&= \frac{6u(x)x - Q(x)h(x)}{2} - 3\sqrt{2}x_0 Q(x)x + x_0^2 h(x) \\
&= 3(u(x) - \sqrt{2}x_0 Q(x))x + (x_0^2 - \frac{Q(x)}{2})h(x)
\end{aligned}
$$

thus (4.22) implies $x^{\#\#} = N(\mathbf{x})x$. Combining this with (4.21) yields the required adjoint identity $\mathbf{x}^{\#\#} = N(\mathbf{x})\mathbf{x}$ and proves that $N(\mathbf{x})$ is Jordan.

Next, (4.6) follows immediately from the sharp identity (2.15)

$$\mathbf{x}^2 = \mathbf{x}^{\#} + \mathrm{Tr}(\mathbf{x})\mathbf{x} - S(\mathbf{x})\mathbf{e}$$

and $\mathrm{Tr}(\mathbf{x}) = 3x_0$, $S(\mathbf{x}) = 3x_0^2 - \frac{3}{2}Q(x)$.

Finally, in order to finish the proof, we suppose that two eiconal triples (W, Q, u) and $(\tilde{W}, \tilde{Q}, \tilde{u})$ are equivalent, i.e. there exists a linear map $O : (W, Q) \to (\tilde{W}, \tilde{Q})$ such that O is invertible, $\tilde{Q}(Ox) = Q(x)$ for all $x \in W$, and $u(x) = \tilde{u}(Ox)$. The map $\psi : J(u) \to J(\tilde{u})$ by $\psi(x_0, x) = (x_0, O\tilde{x})$ is obviously a isomorphism of $W = \mathbb{F} \times V$ and $\tilde{W} = \mathbb{F} \times \tilde{V}$ on the level of vector spaces. Furthermore, applying (4.6) we obtain

$$\psi(\mathbf{x})^2 = (x_0^2 + \tilde{Q}(Ox),\ 2x_0\,Ox + \frac{1}{3\sqrt{2}}\nabla\tilde{u}(Ox)). \tag{4.23}$$

On the other hand,

$$\partial_y u|_x \equiv \lim_{t \to 0} \frac{u(x + ty) - u(x)}{t} = \lim_{t \to 0} \frac{\tilde{u}(Ox + tOy) - \tilde{u}(Ox)}{t} \equiv \partial_{Oy}\tilde{u}|_{Ox}$$

which yields by the definition of the covariant gradient

$$Q(\nabla u(x); y) = \partial_y u|_x = \partial_{Oy}\tilde{u}|_{Ox} = \tilde{Q}(\nabla\tilde{u}(Ox); Oy) = Q(O^{-1}\nabla\tilde{u}(Ox); y),$$

hence by nondegeneracy of Q, $\nabla u(x) = O^{-1}\nabla\tilde{u}(Ox)$. Hence (4.23) yields

$$\psi(\mathbf{x})^2 = \big(x_0 + Q(x),\ O(2x_0 x + \frac{1}{3\sqrt{2}}\nabla u(x))\big) = \psi(\mathbf{x}^2).$$

The polarization of the obtained identity implies that $\psi(\mathbf{x}\bullet\mathbf{y}) = \psi(\mathbf{x})\bullet\psi(\mathbf{y})$, hence ψ is an isomorphism of the Jordan algebras, as required.

$\square$

Remark 4.1. It also is convenient to exhibit the Jordan multiplicative structure on $J(V, N, e)$ explicitly by polarizing the quadratic identity (4.6), namely

$$\mathbf{x} \bullet \mathbf{y} = (x_0 y_0 + Q(x, y),\ x_0 y + y_0 x + \frac{1}{3\sqrt{2}} h(x; y)), \tag{4.24}$$

where h is defined by (4.8).

To finish the proof of Theorem 1 it suffices to verify the following

Proposition 4.3. *Let $\alpha : \mathcal{J} \to \mathcal{E}$ and $\beta : \mathcal{E} \to \mathcal{J}$ are the correspondences defined in Proposition 4.1 and Proposition 4.2, respectively. Then $\alpha\beta = \mathrm{id}_{\mathcal{E}}$ and $\beta\alpha = \mathrm{id}_{\mathcal{J}}$.*

Proof. (i) We prove first that $\beta\alpha = id_{\mathcal{J}}$. Let $J = J(V, N, e)$ and let $\alpha(J) = (e^{\perp}, \tau, u)$ be the cubic eiconal associated to J by virtue of Proposition 4.1, where $u = \sqrt{2}N|_{e^{\perp}}$. By Proposition 4.2

$$\beta(\alpha(J)) = J(\widehat{V}, \widehat{N}, \mathbf{e}), \qquad \widehat{V} = \mathbb{F} \times e^{\perp},$$

where $\widehat{N}$ is found by (4.4) as

$$\widehat{N}(\mathbf{x}) = x_0^3 - \frac{3}{2}x_0\tau(x) + \frac{1}{\sqrt{2}}\, u(x)$$

$$= x_0^3 - \frac{3}{2}x_0\tau(x) + N(x), \qquad \mathbf{x} = (x_0, x) \in \widehat{V}.$$

We claim that $\phi(\mathbf{x}) = x_0 e + x : \beta(\alpha(J)) \to J$ is an isomorphism of the Jordan algebras. Notice that ϕ is an $\mathbb{F}$-linear map and $\ker \phi = \{\mathbf{0}\}$. Thus, it suffices to verify that ϕ is compatible with the multiplicative structure which is equivalent to showing that $\phi(\mathbf{x}^2) = \phi(\mathbf{x})^2$. To this end we notice that by (4.6) and the definition of u, $\mathbf{x}^2 = (x_0^2 + Q(x), 2x_0x + \frac{1}{3}\nabla N(x))$. On the other hand, by the definition of ϕ,

$$\phi(\mathbf{x})^2 = x_0^2 e + 2x_0 x + x^2,$$

which readily yields

$$\phi(\mathbf{x}^2) - \phi(\mathbf{x})^2 = \frac{1}{3}\nabla N(x) - x^2 + T(x)e, \tag{4.25}$$

where the gradient is found by (3.2) as follows: $\nabla N(x) = (DN(x))^{e^{\perp}} = 3(x^{\#})^{e^{\perp}}$. Furthermore, using $\mathrm{Tr}(x) = 0$ and (2.15), we obtain

$$(x^{\#})^{e^{\perp}} = (x^2 - \mathrm{Tr}(x)x + S(x)e)^{e^{\perp}} \equiv (x^2)^{e^{\perp}}$$

$$= x^2 - T(x^2; e)e = x^2 - T(x; x)e$$

$$= x^2 - T(x)e.$$

Thus (4.25) yields $\phi(\mathbf{x}^2) - \phi(\mathbf{x})^2 = 0$, which implies the desired isomorphy of ϕ.

(ii) In the converse direction, let us consider an eiconal triple $\epsilon = (W, Q, u)$ and associate to it the Jordan algebra $\beta(\epsilon) = J(V, N, \mathbf{e})$, where $V = \mathbb{F} \times W$, N is defined by (4.4), and $\mathbf{e} = (1, 0)$. By (4.5), $\mathbf{e}^{\perp} = 0 \times W$, and also by (4.4) and (4.5), $N(\mathbf{x}) = \frac{u(x)}{\sqrt{2}}$ and $\tau(\mathbf{x}) = Q(x)$ for $\mathbf{x} = (0, x) \in \mathbf{e}^{\perp}$. Then Proposition 4.1 yields:

$$\alpha\beta(\epsilon) = (0 \times W, Q, u)$$

which is trivially equivalent to ϵ. The proposition is proved.

$\square$

References

[1] L. Borsten, D. Dahanayake, M. J. Duff, H. Ebrahim, and W. Rubens. Black holes, qubits and octonions. *Phys. Rep.*, 471(3-4):113–219, 2009.

[2] P.-E. Chaput. Severi varieties. *Math. Z.*, 240(2):451–459, 2002.

[3] P.-E. Chaput. Scorza varieties and Jordan algebras. *Indag. Math. (N.S.)*, 14(2):169–182, 2003.

[4] P. Etingof, D. Kazhdan, and A. Polishchuk. When is the Fourier transform of an elementary function elementary? *Selecta Math. (N.S.)*, 8(1):27–66, 2002.

[5] J. Faraut and A. Korányi. *Analysis on symmetric cones*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1994. Oxford Science Publications.

[6] N. Jacobson. Generic norm of an algebra. *Osaka Math. J.*, 15:25–50, 1963.

[7] N. Jacobson. *Structure and representations of Jordan algebras*. American Mathematical Society Colloquium Publications, Vol. XXXIX. American Mathematical Society, Providence, R.I., 1968.

[8] P. Jordan. Über verallgemeinerungsmöglichkeiten des formalismus der quantenmechanik. *Nachr. Akad. Wiss. Göttingen. Math. Phys.*, 41:209–217, 1933.

[9] P. Jordan, J. von Neumann, and E. Wigner. On an algebraic generalization of the quantum mechanical formalism. *Ann. of Math. (2)*, 35(1):29–64, 1934.

[10] S. Krutelevich. Jordan algebras, exceptional groups, and Bhargava composition. *J. Algebra*, 314(2):924–977, 2007.

[11] K. McCrimmon. *A taste of Jordan algebras*. Universitext. Springer-Verlag, New York, 2004.

[12] N. Nadirashvili, V.G. Tkachev, and S. Vlăduţ. A non-classical solution to a Hessian equation from Cartan isoparametric cubic. *Adv. Math.*, 231(3-4):1589–1597, 2012.

[13] L. Pirio and F. Russo. Extremal varieties 3-rationally connected by cubics, quadro-quadric cremona transformations and rank 3 jordan algebras. *arXiv:1109.3573*, 2011.

[14] L. Pirio and F. Russo. Quadro-quadric cremona transformations in low dimensionsv via the jc-correspondence. *arXiv:1204.0428*, 2012.

[15] T. A. Springer. On a class of Jordan algebras. *Nederl. Akad. Wetensch. Proc. Ser. A 62 = Indag. Math.*, 21:254–264, 1959.

[16] V. G. Tkachev. A generalization of Cartan's theorem on isoparametric cubics. *Proc. Amer. Math. Soc.*, 138(8):2889–2895, 2010.