manuscripta math. 112, 519–538 (2003)

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Length functions of lemniscates

Received: 24 March 2003 / Revised version: 11 July 2003
Published online: 4 November 2003

Abstract. We study metric and analytic properties of generalized lemniscates $E_t(f) = \{z \in \mathbb{C} : \ln|f(z)| = t\}$, where f is an analytic function. Our main result states that the length function $|E_t(f)|$ is a bilateral Laplace transform of a certain positive measure. In particular, the function $\ln|E_t(f)|$ is convex on any interval free of critical points of $\ln|f|$. As another application we deduce explicit formulae of the length function in some special cases.

1. Introduction

Throughout this paper $E_f(t)$ denotes the t-level set

$$\ln|f(z)| = t \tag{1}$$

of an analytic function $f(z)$.

Our starting point is the polynomial lemniscates. Let $f(z)$ be a monic polynomial $P(z) = z^n + a_1 z^{n-1} + \ldots + a_n$, $n \geq 2$. In 1958 Erdös, Herzog and Piranian [13] posed a number of problems concentrated around the metric properties of lemniscates (see also the later paper [12]). Among them is the following

Erdös Conjecture (Problem 12, [13]; Problem VI, [12])**.** *For fixed degree n of P, is the length of the lemniscate $|P(z)| = 1$ greatest in the case where $P(z) = Q_n(z) := z^n - 1$? Is the length at least 2π, if $E_P(0)$ is connected?*

The actual breakthrough in the Erdös conjecture was made recently by A. Eremenko and W. Hayman in [14]. They proved that for any degree n there

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The author was supported by Russian President grant for young doctorates no. 00-15-99274 and grant RFBR no. 03-01-00304.

Mathematics Subject Classification (2000): 30E05, 42A82, 44A10

DOI: 10.1007/s00229-003-0411-3

exist a polynomial $P^*(z)$ which maximizes the length of $E_P(0)$ (following [14], we call it an *extremal* polynomial); moreover, the estimate

$$c_n \equiv \max_{\deg P=n} |E_P(0)| \leq An$$

holds, where $|E|$ denotes the length of E, and $A \approx 9.173$. One can readily check that the conjectural value is $|E_{Q_n}(0)| = 2n + O(1)$. The previous upper estimates were due to Ch. Pommerenke [28]: $c_n \leq 74n^2$ and P. Borwein [6]: $c_n \leq 8\pi en$.

Another important results of [14] states that *the lemniscate $E_{P^*}(0)$ is always connected and for any degree n there exists an extremal polynomial P^* such that all its critical points belong to $E_{P^*}(0)$.*

Concerning the first part of Erdös conjecture, which is still unsolved, T. Erdelyi writes that "this problem seems almost impossible to settle" [11, p. 8]. Another difficulty in the study of the problem is the absence of any explicit formulae for the *length function* $|E_P(t)|$ (except for the trivial polynomials $P(z) = (z-a)^n$). This question was initially posed by Piranian in [26] for the rose-type polynomials Q_n; explicit formulae of $|E_{Q_n}(t)|$ were obtained by Butler [7] and Elia [9].

The second part of Erdös conjecture is related to the lower estimate of $|E_P(0)|$ for so-called K-polynomials, i.e. the polynomials with connected lemniscate $E_P(0)$. This problem was solved in affirmative by Pommerenke in [27], who established that

$$\min_{P\in K,\ \deg P=n} |E_P(0)| = |E_{(z-a)^n}(0)| = 2\pi. \tag{2}$$

1.1. Main results

Definition 1. *By a lemniscate region of f we mean a triple $(\mathcal{U}, f, \mathfrak{I})$ where $f(z)$ is an analytic function, $\mathcal{U}$ is a component of the set*

$$\mathcal{U}_f(\mathfrak{I}) := \{z \in \mathbb{C} : a < \ln|f(z)| < b\},$$

$\mathfrak{I} = (a, b)$, such that for every $t \in \mathfrak{I}$ the set

$$E_{\mathcal{U},f}(t) := E_f(t) \cap \mathcal{U}$$

is compact in $E_f(t)$. A lemniscate region will called regular if $\mathcal{U}$ contains no zeroes and no critical points, i.e. $f(z)f'(z) \neq 0$ in $\mathcal{U}$ (cf. [18, p. 264]).

It is easy to see that any (analytic) function f admits regular lemniscate regions provided that the set of zeroes

$$\mathcal{Z}(f) := \{z : \ f(z) = 0\}$$

is nonempty. Disregarding the polynomial case, where all lemniscates are compact curves, we mention, for instance, an example of a regular lemniscate region (the corresponding lemniscate family is drawn on Figure 1)

$$f(z) = \sin z, \qquad \mathfrak{I} = (-\infty, 0), \qquad \mathcal{U} = \{z : 0 < |\sin z| < 1, |\operatorname{Re} z| < \frac{\pi}{2}\}.$$

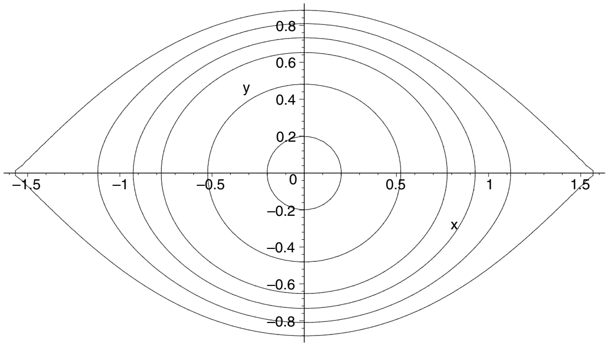

Fig. 1. The regular lemniscate region for $f(z) = \sin z$

By its definition, the projection $\ln|f(z)| : \mathcal{U} \to \mathfrak{I}$ is a proper map. Hence, in case of a regular lemniscate region, the level set $E_{\mathcal{U},f}(t)$ splits into a finite collection of simple closed curves; any finite collection of closed components of $E_{\mathcal{U},f}(t)$ will be called a *t-lemniscate*, or just a lemniscate of f.

Given a lemniscate domain $(\mathcal{U}, f, \mathfrak{I})$ we define the corresponding *length function*

$$|E_{\mathcal{U},f}(t)| := \operatorname{length}(E_{\mathcal{U},f}(t)), \quad t \in \mathfrak{I}, \tag{3}$$

which will be in focus of the present paper. On the other hand, the expressions for the higher derivatives of $|E_{\mathcal{U},f}(t)|$ (see (22) below) involve integrals of the kind

$$L_w(t) = \|w\|_t^2 := \int\limits_{E_{\mathcal{U},f}(t)} |w(z)|^2 \, |dz|, \tag{4}$$

where $w(z)$ is an analytic in $\mathcal{U}$ function. This makes it natural to consider the averages (4) as a suitable generalization of the length function. The case $w \equiv 1$ obviously reduces to the length function.

Further, we consider the following first-order differential operator

$$G_f(w) = w_{[1]} := 2gw' + g'w, \qquad w_{[k]} = G_f^k(w), \tag{5}$$

where $g = f/f'$, and $w_{[k]}$ are the G_f-iterations of $w \equiv w_{[0]}$.

The following theorem summarizes the results given in Section 2.2 below.

Theorem 1. *Let $(\mathcal{U}, f, \mathfrak{I})$ be a regular lemniscate region and $w(z)$ be an analytic in $\mathcal{U}$ function. Then $L_w(t) \in C^\infty(\mathfrak{I})$. Moreover, for any $t \in \mathfrak{I}$ the sequence of derivatives $L_w^{(k)}(t)$, $k \geq 0$, $t \in \mathfrak{I}$, forms a Hamburger moment sequence; that is, for any $t \in \mathfrak{I}$ and for all $p = 0, 1, \ldots,$ the Hankel matrices*

$$\begin{pmatrix} L_w(t) & L'_w(t) & \dots & L_w^{(p)}(t) \\ L'_w(t) & L''_w(t) & \dots & L_w^{(p+1)}(t) \\ \vdots & \vdots & \ddots & \vdots \\ L_w^{(p)}(t) & L_w^{(p+1)}(t) & \dots & L_w^{(2p)}(t) \end{pmatrix} \tag{6}$$

are non-negative definite. Moreover, the entries of the latter matrix are the following scalar products

$$L_w^{(k)}(t) = \langle w_{[j]}; w_{[k-j]}\rangle_t := \int\limits_{E_{\mathcal{U},f}(t)} \operatorname{Re} \overline{w}_{[k-j]} w_{[j]} \, |dz|, \qquad 0 \le j \le k \tag{7}$$

where the integrals are independent of the choice of j.

Now, applying the well-known Bernstein's theorem [5] (see, also [30, Ch. VI]), we obtain a bilateral Laplace representation of L_w

Corollary 1. *Let $(\mathcal{U}, f, \mathfrak{I})$ be a regular lemniscate region. Given an analytic in $\mathcal{U}$ function $w(z)$ there exists a non-decreasing function $\sigma(x)$ such that*

$$L_w(t) = \int\limits_{E_{\mathcal{U},f}(t)} |w(z)|^2 \, |dz| = \int\limits_{-\infty}^{+\infty} e^{xt} \, d\sigma(x), \qquad t \in (\alpha; \beta), \tag{8}$$

and the latter integral converges for $t \in \mathfrak{I}$.

Functions $L(t)$ which satisfy a bilateral Laplace representation (8) are known as *exponentially convex* functions ([5], [1, § V.5.4]). This means that the associated with $L(t)$ stationary kernel $\mathbf{L}(x, y) = L(\frac{x+y}{2})$ is of positive type, i.e. for every finite sequence $\{t_j\}_1^m$ from $\mathfrak{I}$ the quadratic form

$$\sum_{i,j=1}^{m} L\left(\frac{t_i + t_j}{2}\right) \xi_i \xi_j$$

is positive (definite or semidefinite). In particular, given an exponentially convex function $L(t)$ the function $\ln L(t)$ is convex.

This class was introduced and extensively studied by S. Bernstein [5] and D. Widder [30] in connection with the so-called completely (or absolutely) monotonic analytic functions (see the definition in Section 3.2). We only mention a deep penetration of the both classes into complex analysis, inequalities analysis [2], special functions [25], probability theory [19], radial-function interpolation [29], harmonic analysis on semigroups [3] (for further discussion and references, see recent survey [4]).

We also remark that exponential convexity leads to further inequalities on L_w and its derivatives like those considered in [16].

As another consequence of exponential convexity (see [5, § 15]) we point out the following continuation property, actually, a *complexification* of the length function (see also (53) below)

Corollary 2. *Under the hypotheses of Theorem 1, the function $L_w(t)$ admits an analytic continuation $L_w(z)$ into the strip $a < \operatorname{Re} z < b$, where $\mathfrak{I} = (a, b)$.*

In the remaining part of this section we consider the case when the function $f(z)$ is a monic polynomial $P(z)$ and $w(z) \equiv 1$. By $T_1 < \cdots < T_{\nu-1}$ we denote the set of all (finite) critical values of $\ln |P|$. The intervals $\mathfrak{I}_j = (T_{j-1}, T_j)$ will be called *regular* (with respect to P), where $T_0 = -\infty$, $T_\nu = +\infty$. Then $(\mathcal{U}_P(\mathfrak{I}_j), P, \mathfrak{I}_j)$, $1 \le j \le \nu - 1$, constitutes a special class of the *principal* regular lemniscate regions.

We have in the preceding notations

Corollary 3. *Given a regular interval $\mathfrak{I}_j = (T_{j-1}, T_j)$, $1 \le j \le \nu$, the following representation holds*

$$|E_P(t)| = \int_{-\infty}^{+\infty} e^{xt}\, d\sigma^{P,\mathfrak{I}_j}(x), \qquad t \in \mathfrak{I}_j, \tag{9}$$

where $\sigma^{P,\mathfrak{I}_j}(x)$ is a non-decreasing function. In particular, $\ln |E_P(t)|$ *is a convex function on $\mathfrak{I}_j$.*

It turns out (see Section 3.2 below) that the function $\sigma^{P,\mathfrak{I}_\nu}(x)$ (i.e., $j = \nu$) is a piece-wise constant function and the integral in (9) can be written as sum of a certain exponential series.

Given a monic polynomial P, $\deg P = n$, we define an auxiliary function

$$\Phi_P(t) := \ln |E_P(t)| - \frac{t}{n}.$$

We call $\Phi_P(t)$ the *indicator* of P. Then $\Phi_P(t)$ has a simple invariance property with respect to dilatations of P

$$P_\alpha(z) := e^{-\alpha} P(z e^{\frac{\alpha}{n}}) = z^n + a_1 e^{-\frac{\alpha}{n}} z^{n-1} + \cdots + a_{n-1} e^{-\frac{\alpha(n-1)}{n}} z + a_n e^{-\alpha} \tag{10}$$

where $\alpha \in (-\infty; +\infty]$. Indeed, we note that $P_\alpha(z)$ is also a monic polynomial, $P_0(z) \equiv P(z)$, and

$$E_{P_\alpha}(\beta) = e^{-\frac{\alpha}{n}} E_P(\alpha + \beta). \tag{11}$$

Hence, we have

$$\Phi_{P_\alpha}(\beta) = \Phi_P(\alpha + \beta). \tag{12}$$

Proposition 1. *Let $P^*(z)$ be an extremal (with respect to Erdös conjecture) polynomial of degree n; then $t = 0$ is an absolute maximum point of $\Phi_{P^*}(t)$.*

Proof. It follows from (12) that

$$\Phi_{P^*}(t) = \Phi_{P_t^*}(0) = |E_{P_t^*}(0)| \le |E_{P^*}(0)| = \Phi_{P^*}(0)$$

which proves the required property. $\square$

Theorem 2. *$\Phi_P(t)$ and $|E_P(t)|$ are continuous functions in $(-\infty; +\infty)$. Moreover, if the polynomial $P(z)$ is non-trivial (i.e. is different from $(z-a)^n$) then*

A) $\Phi_P(t)$ and $|E_P(t)|$ are strictly convex in each regular interval (T_{j-1}, T_j), $1 \leq j \leq \nu$;
B) the following asymptotics behavior holds

$$\lim_{t \to +\infty} \Phi_P(t) = \ln 2\pi. \tag{13}$$

Remark 1. A direct analysis near critical points of $\ln |P|$ implies that $\Phi_P(t)$ is only of Hölder class there which makes useless the standard variational methods near the corresponding extremum.

As another consequences of Theorem 2 we mention the estimate (2) due to Pommerenke.

Corollary 4. *Let P be a monic K-polynomial, i.e. the lemniscate $E_P(0)$ is connected. Then*

$$|E_P(0)| \geq 2\pi,$$

with equality only in the case $P(z) = (z-a)^n$.

Proof. We can assume that $P(z) \neq (z-a)^n$, otherwise $|E_P(0)| = 2\pi$. Then it easily follows from the definition of K-polynomial that $T_k \leq 0$ for all $k \leq \nu$. Hence, by virtue of Theorem 2 we conclude that $\Phi_P(t)$ is strictly convex in $[0; +\infty)$. Moreover, by (13) the function $\Phi_P(t)$ is bounded on $[0; +\infty)$ and it follows from the strict convexity of $\Phi_P(t)$ that it is actually strictly decreasing. Because of $\Phi_P(0) = \ln |E_P(0)|$, we have $\Phi_P(0) > \ln 2\pi$, or $|E_P(0)| > 2\pi$, which completes the proof. □

By the Eremenko-Hayman theorem, we know that for all integers $n \geq 2$ an extremal polynomial P^*, $\deg P = n$, does exist such that $\mathcal{Z}(P^{*\prime}) \subset E_P(0)$. The following assertion gives a complement to the latter property.

Corollary 5. *If $P^*(z)$ is an extremal polynomial of degree n then the lemniscate $E_{P^*}(0)$ is singular, i.e. it contains at least one critical point:*

$$\mathcal{Z}(P^{*\prime}) \cap E_{P^*}(0) \neq \varnothing.$$

Proof. By Proposition 1, $t = 0$ is an absolute maximum of the indicator function $\Phi_{P^*}(t)$ and it follows that $t = 0$ can not be a regular value of $\ln |P(z)|$ for $\Phi_{P^*}(t)$ is strictly convex in a neighborhood of regular values. □

As another application of (7), in Section 3.1 we obtain explicit formulae for the length functions in the case when f is a solution of the following equation

$$\varphi' = C(1-\varphi^\nu)^{\frac{k+1}{\nu}}.$$

2. Proof of the main results

2.1. Preliminaries

Here we prove the main technical result which we formulate in a form suitable for further applications.

Let M be a p-dimensional Riemannian manifold and by $\langle X; Y\rangle$ and ∇ the intrinsic scalar product and covariant derivative are denoted. By div X we denote the divergence of a vector field X generated by ∇. We recall, that a function $u(x) : M \to \mathbb{R}$ is called *harmonic* if $\Delta u \equiv \operatorname{div} \nabla u(x) = 0$; by $\Sigma_u(t)$ we denote the level set $\{x \in M : u(x) = t\}$.

Definition 2. *A triple $(\mathcal{U}, u, \mathfrak{I})$, where $\mathcal{U}$ is an open subset, $\overline{\mathcal{U}} \subset M$, $u(x)$ is a harmonic function in $\mathcal{U}$, and $\mathfrak{I} = (\alpha; \beta)$, is said to be a (regular) lemniscate region if for all $t \in \mathfrak{I}$ the set $\Sigma_{\mathcal{U},u}(t) := \Sigma_u(t) \cap \mathcal{U}$ is compactly contained in $\mathcal{U}$ and $\mathcal{U}$ is free of critical points of u.*

Remark 2. Clearly, when $M = \mathbb{C}$ the preceding definition is reduces to the that one given in Section 1.1. We will not distinguish the corresponding notations $(\mathcal{U}, f, \mathfrak{I})$ and $(\mathcal{U}, \ln|f|, \mathfrak{I})$, when f is an analytic function; moreover, in this case

$$E_f(t) = \Sigma_{\ln|f|}(t), \qquad E_{\mathcal{U},f}(t) = \Sigma_{\mathcal{U},\ln|f|}(t).$$

Lemma 1. *Let $(\mathcal{U}, u, \mathfrak{I})$ be a lemniscate region and $h(x)$ be a C^2-smooth function on $\mathcal{U}$ and*

$$H(t) := \int\limits_{\Sigma_{\mathcal{U},u}(t)} h(x)|\nabla u(x)|\, d\mathcal{H}^{p-1}(x), \tag{14}$$

where $d\mathcal{H}^{p-1}$ denotes $(p-1)$-dimensional Hausdorff measure on $\Sigma_{\mathcal{U},u}(t)$.

Then $H(t) \in C^2(\mathfrak{I})$ and

$$H'(\tau) = \int\limits_{\Sigma_{\mathcal{U},u}(t)} \frac{\langle \nabla h(x); \nabla u(x)\rangle}{|\nabla u(x)|}\, d\mathcal{H}^{p-1}(x), \tag{15}$$

$$H''(\tau) = \int\limits_{\Sigma_{\mathcal{U},u}(t)} \frac{\Delta h(x)}{|\nabla u(x)|}\, d\mathcal{H}^{p-1}(x). \tag{16}$$

Proof. Because of regularity condition all the level sets $\Sigma_{\mathcal{U},u}(t)$ are embedded submanifolds in M and the vector field

$$\nu(x) \equiv \frac{\nabla u(x)}{|\nabla u(x)|} \tag{17}$$

represents the field of unit normals to $\Sigma_{\mathcal{U},u}(t)$ pointing in the growth direction of u,

$$\langle \boldsymbol{\nu}(x); \nabla u(x)\rangle = |\nabla u(x)|. \tag{18}$$

We claim that for any C^1-vector field $\mathbf{v}$ on $\mathcal{U}$

$$\frac{d}{d\tau}\int\limits_{\Sigma_{\mathcal{U},u}(\tau)} \langle \mathbf{v};\,\nu\rangle\, d\mathcal{H}^{p-1} = \int\limits_{\Sigma_{\mathcal{U},u}(\tau)} \frac{\operatorname{div}\mathbf{v}}{|\nabla u|}\, d\mathcal{H}^{p-1}. \tag{19}$$

Indeed, let $t\in\mathfrak{I}$, $t\neq\tau$, be chosen arbitrary. Then by virtue of (18) and harmonicity of $u(x)$ we have by Stokes' formula

$$\begin{aligned} F(t)-F(\tau) &= \int\limits_{\mathcal{U}(t)-\mathcal{U}(\tau)} \langle \mathbf{v};\,\nu\rangle\, d\mathcal{H}^{p-1} \\ &= \int\limits_{\partial\mathcal{U}(\tau,t)} \langle \mathbf{v};\,\nu\rangle\, d\mathcal{H}^{p-1} = \int\limits_{\mathcal{U}(t)-\mathcal{U}(\tau)} \operatorname{div}\mathbf{v}\; dx, \end{aligned} \tag{20}$$

where $\mathcal{U}(t)=\{x\in\mathcal{U} : u(x)<t\}$ and $F(\tau)$ denotes the left-hand side integral in (19).

Then applying co-area formula to (20) we obtain

$$\frac{F(t)-F(\tau)}{t-\tau} = \frac{1}{t-\tau}\int\limits_{\tau}^{t} d\xi \int\limits_{\Sigma_{\mathcal{U},u}(\xi)} \frac{\operatorname{div}\mathbf{v}}{|\nabla u|}\, d\mathcal{H}^{p-1}. \tag{21}$$

The latter limit does exist for every regular value τ of $u(x)$ (even if u is only locally Lipschitz in D [15, § 3.2]) and (19) follows.

Thus, applying (19) to $\mathbf{v}=h\nabla u$ we obtain (15). Moreover, it follows from (17) that (15) can be written in the form

$$H'(\tau) = \int\limits_{\Sigma_{\mathcal{U},u}(\tau)} \langle \nabla h;\,\nu\rangle\, d\mathcal{H}^{p-1}$$

and, again applying (19) to the last relation, now with $\mathbf{v}=\nabla h(x)$ we arrive at

$$H''(\tau) = \int\limits_{\Sigma_{\mathcal{U},u}(\tau)} \frac{\operatorname{div}\nabla h}{|\nabla u|}\, d\mathcal{H}^{p-1}$$

and the lemma is proved. □

Remark 3. We notice that in the case when M is a minimal submanifold of $\mathbb{R}^N$ and $u(x)$ is a coordinate function on M, the definition of a regular lemniscate region corresponds to a special class of minimal surfaces, so-called minimal tubes, in Euclidean space [23], [24]. In that case a result similar to Lemma 1 (actually, for the radial symmetric functions h) was obtained by V. Klyachin in [20].

The following assertion is an easy consequence of Cauchy's inequality and the last theorem, and it can be regarded as a particular case of Theorem 1 for $p=1$.

Corollary 6. *Let* $\Delta\ln h\geq 0$*; then, under the hypotheses of Lemma 1, the function* $\ln H(t)$ *is convex in* $\mathfrak{I}$.

2.2. Representations of L_w

Here and in what follows we use the notations of Section 1.1.

Lemma 2. *Let $(\mathcal{U}, f, \mathfrak{I})$ be a regular lemniscate region and $w = w_{[0]}$ be an analytic function in $\mathcal{U}$. Let $L_w(t) = \|w\|_t^2$ (see (4)); then $L_w(t) \in C^\infty(\mathfrak{I})$ and for any $\nu \geq 0$*

$$L_w^{(2\nu+1)}(t) = \int\limits_{E_{\mathcal{U},f}(t)} \operatorname{Re} \overline{w}_{[\nu]} w_{[\nu+1]} \, |dz|,$$
$$L_w^{(2\nu+2)}(t) = \int\limits_{E_{\mathcal{U},f}(t)} |w_{[\nu+1]}|^2 \, |dz|, \tag{22}$$

where $w_{[k]}(z)$ are defined by (5).

Proof. To apply Lemma 1 (see also Remark 2) we note that in our case $M = \mathbb{C}$. Let us identify a complex number $z = x + iy$ with the point $(x, y) \in \mathbb{R}^2$ so that the gradient of a real-valued function $h(x, y)$ takes the form $h'_x + ih'_y$. Then by Cauchy-Riemann theorem we have for any analytic function $F(z)$

$$\nabla \operatorname{Re} F(z) \equiv (\operatorname{Re} F(z))'_x + i(\operatorname{Re} F(z))'_y = \overline{F'(z)}, \tag{23}$$

whence

$$\overline{\nabla u(z)} = \overline{\nabla \operatorname{Re} \ln f(z)} = \frac{f'(z)}{f(z)} := \frac{1}{g(z)}. \tag{24}$$

Here and in the sequel we use the logarithms only for brevity of the computations of gradients and, of course, one can deduce these formulae directly. Henceforth, we chose the main branch of logarithm, $\ln 1 = 0$.

Moreover, we point out that $g(z)$, being defined by (24), is an analytic function in $\mathcal{U}$ because of the regularity condition: $f'(z) \neq 0$. As a consequence, the same property is obviously true for the iterates $w_{[k]}$, $k \geq 0$.

Now, let $t \in \mathfrak{I}$. Applying (24) we get for $w = w_{[0]}$

$$L_w(t) = \int\limits_{E_{\mathcal{U},f}(t)} \frac{|w(z)|^2}{|\nabla u(z)|} |\nabla u(z)| \, |dz| = \int\limits_{E_{\mathcal{U},f}(t)} |w^2(z) g(z)| \, |\nabla u(z)| \, |dz|.$$

Hence, substituting $h(z) = |w^2(z)g(z)|$ in (15) yields

$$L'_w(t) = \int\limits_{E_{\mathcal{U},f}(t)} \frac{\langle \nabla h(z); \nabla u(z) \rangle}{|\nabla u(z)|} \, |dz|. \tag{25}$$

Further, we note that $\ln h \equiv \operatorname{Re} \ln w^2 g$, and it follows from (23) that

$$\overline{\nabla h}(z) = h(z) \frac{d}{dz} (\ln w^2(z) g(z)) = \frac{|w^2 g|}{wg} w_{[1]} = \overline{w} w_{[1]} \frac{|g|}{g}. \tag{26}$$

On the other hand, we obtain from (24) and (26)

$$\langle \nabla u; \nabla h\rangle = \operatorname{Re}\left(\overline{\nabla u}\nabla h\right) = \frac{1}{|g|}\operatorname{Re}\overline{w}w_{[1]}. \tag{27}$$

Substituting (27) and (24) in (25) yields

$$L'_w(t) = \int\limits_{E_{\mathcal{U},f}(t)} \operatorname{Re}\overline{w}(z)w_{[1]}(z)\,|dz|. \tag{28}$$

To find the second derivative $L''_w(t)$ we notice that $\ln h(z)$ is a harmonic function. It follows then

$$0 = \Delta \ln h(z) = \frac{\Delta h(z)}{h(z)} - \frac{|\nabla h(z)|^2}{h^2(z)},$$

and by (26) we arrive at

$$\Delta h(z) = \frac{1}{h(z)}|\nabla h(z)|^2 = \frac{|ww_{[1]}|^2}{|w^2 g|} = \frac{|w_{[1]}|^2}{|g|}.$$

After substituting the above expressions in (16) and taking into account that $\nabla u = 1/|g|$ we obtain from (24)

$$L''_w(t) = \int\limits_{E_{t,\mathcal{U}}(f)} |w_{[1]}|^2\,|dz|. \tag{29}$$

Now we observe that, by virtue of the regularity condition, w_1 and the consequent iterates w_ν, $\nu \geq 2$, are analytic functions in $\mathcal{U}$. Since the integral in $L''_w(t)$ takes the form (4), it is clear now that formulae (22) can be obtained by induction from (28) and (29), and the lemma follows. $\square$

The further properties of $\|w\|_t^2$ can be deduced by using the following formalism. Let $(\mathcal{U}, f, \mathfrak{I})$ be a regular lemniscate region. We endow the space of all analytic in $\mathcal{U}$ functions by a family of the following scalar products

$$\langle u; v\rangle_t := \int\limits_{E_{t,\mathcal{U}}(f)} \operatorname{Re}\overline{u(z)}v(z)\,|dz|$$

and $\|w\|_t^2 := \langle w; w\rangle_t$. Thus, (22) can be rewritten as

$$D^{2\nu}\|w\|_t^2 = \|w_{[\nu]}\|_t^2, \tag{30}$$

$$D^{2\nu+1}\|w\|_t^2 = \langle w_{[\nu]}; w_{[\nu+1]}\rangle_t \tag{31}$$

where $D = \frac{d}{dt}$.

We can polarize the preceding identities in the standard way by using linearity of D. Let $\nu = 0$ in (31); then substituting the sum $v_{[0]} + u_{[0]}$ for $w_{[0]}$ yields

$$\begin{aligned} D\|u_{[0]} + v_{[0]}\|_t^2 &= \langle u_{[0]} + v_{0]}; u_{[1]} + v_{[1]}\rangle_t \\ &= \langle u_{[0]}; u_{[1]}\rangle_t + \langle v_{[0]}; v_{[1]}\rangle_t \\ &\quad + \langle u_{[0]}; v_{[1]}\rangle_t + \langle v_{[0]}; u_{[1]}\rangle_t \quad \text{(by (30))} \\ &= D\|u_{[0]}\|_t^2 + D\|v_{[0]}\|_t^2 + \langle u_{[0]}; v_{[1]}\rangle_t + \langle u_{[1]}; v_{[0]}\rangle_t, \end{aligned}$$

and it follows that

$$2D\langle u_{[0]}; v_{[0]}\rangle_t = \langle u_{[0]}; v_{[1]}\rangle_t + \langle u_{[1]}; v_{[0]}\rangle_t \tag{32}$$

holds for two any analytic functions in $\mathcal{U}$.

Corollary 7. *Let* $(\mathcal{U}, f, \mathfrak{I})$ *be a regular lemniscate region and* $w = w_{[0]}$ *be an analytic function in* $\mathcal{U}$. *Then for all* $n \geq 0$

$$D^n \|w_{[0]}\|_t^2 = \langle w_{[j]}; w_{[n-j]}\rangle_t, \quad \forall j : 0 \leq j \leq n, \qquad t \in \mathfrak{I}, \tag{33}$$

where $w_{[k]} = G_f^k(w)$.

Proof. First we note that by (30)-(31) it is enough to prove that the scalar product in (33) is independent of j. First, applying $k = 0$ to (31), we obtain

$$D\langle w_{[0]}; w_{[0]}\rangle_t = \langle w_{[0]}; w_{[1]}\rangle_t \tag{34}$$

which coincides with (33) for $n = 1$, $j = 0, 1$.

Next, applying $2D$ to both sides of (34) we find by (32) that

$$2D^2\langle w_{[0]}; w_{[0]}\rangle = 2D\langle w_{[0]}; w_{[1]}\rangle_t = \langle w_{[1]}; w_{[1]}\rangle_t + \langle w_{[0]}; w_{[2]}\rangle_t, \tag{35}$$

hence taking into account (30), $k = 1$, we obtain

$$\langle w_{[1]}; w_{[1]}\rangle_t = \langle w_{[0]}; w_{[2]}\rangle_t.$$

The last identity shows that (33) is fulfilled for $n = 2$ and $0 \leq j \leq 2$.

At the remaining part of the proof we apply induction over index n. Namely, we suppose that the statement of our assertion holds for some $n = m \geq 2$. Then applying (33) for $n = m - 1$ we obtain for $u_{[0]} := w_{[1]}$

$$\begin{aligned} D^{m-1}\ D^2\|w_{[0]}\|_t^2 = D^{m-1}\|w_{[1]}\|_t^2 &= D^{m-1}\|u_{[0]}\|_t^2 \\ &= \langle u_{[0]}; u_{[m-1]}\rangle_t = \langle u_{[i]}; u_{[m-i-1]}\rangle_t, \end{aligned}$$

for all $0 \leq i \leq m - 1$. Thus, returning to $w_{[0]}$ we get

$$D^{m+1}\|w_{[0]}\|_t^2 = \langle w_{[1]}; w_{[m]}\rangle_t = \langle w_{[2]}; w_{[m-1]}\rangle_t = \langle w_{[i+1]}; w_{[m-i]}\rangle_t$$

for the same range of i. This proves (33) for $n = m + 1$ and $1 \leq j \leq m$.

On the other hand, applying again the induction assumption $n = m$ to (33) we can write

$$\langle w_{[0]}; w_{[m]}\rangle_t = \langle w_{[1]}; w_{[m-1]}\rangle_t$$

hence applying $2D$ to both sides of the last relation we arrive at

$$\langle w_{[1]}; w_{[m]}\rangle_t + \langle w_{[0]}; w_{[m+1]}\rangle_t = \langle w_{[2]}; w_{[m-1]}\rangle_t + \langle w_{[1]}; w_{[m]}\rangle_t,$$

or $\langle w_{[0]}; w_{[m+1]}\rangle_t = \langle w_{[2]}; w_{[m-1]}\rangle_t$, which yields (33) for the remaining cases $j = 0$ and $j = m + 1$. The corollary is proved completely. $\square$

We recall the following well-known definition (see, e.g., [1, Ch. 2]).

Definition 3. *A sequence (s_k), $k = 0, 1, \dots$ is said to be positive if for any polynomial $P(z) = a_0 + a_1 x + \cdots + a_m x^m$ with real coefficients which is non-negative on $\mathbb{R}$ it holds*

$$a_0 s_0 + a_1 s_1 + \cdots + a_m s_m \geq 0.$$

An equivalent definition of positivity of (s_k) is that the quadratic forms

$$\sum_{i,j=0}^{m} s_{i+j} \xi_i \xi_j$$

should be positive (semidefinite) for all $m \geq 0$ [30, p. 133]. If all the last forms are strictly positive the sequence (s_k) is called a *strictly* positive sequence.

We recall also the following well-known result of H. Hamburger [17] (see also [30, p. 129]).

Hamburger Theorem. *A necessary and sufficient condition that there exist at least one non-decreasing function $\sigma(x)$ such that*

$$\int_{-\infty}^{+\infty} x^k \, d\sigma(x) = s_k, \qquad (k = 0, 1, \dots)$$

with all the integrals converging, is that the sequence (s_k) should be positive.

Now Theorem 1 follows from the property formulated below.

Corollary 8. *Let $(\mathcal{U}, f, \mathfrak{I})$ be a regular lemniscate region and w be an analytic function in $\mathcal{U}$. Then for all $t \in \mathfrak{I}$ the sequence*

$$L_w^{(k)}(t) = D^k \|w\|_t^2, \qquad k = 0, 1, \dots$$

forms a positive sequence. Moreover, $(L_w^{(k)}(t))_{k \geq 0}$ forms a strictly positive sequence (for all $t \in \mathfrak{I}$) if and only if the system of iterates $\{w_{[k]}\}_{k \geq 0}$ is linearly independent.

Proof. It follows from (33) that the corresponding Hankel matrices (6) have the form of Gram matrices. Hence, $(L_w^{(k)}(t))_{k \geq 0}$ is a positive sequence by characteristic property of Gram matrices. The latter assertion of the corollary now can be proved as follows. Let the sequence $L_w^{(k)}(t_0)$ fails strict positivity at some $t_0 \in \mathfrak{I}$; it means that there exists an index $N \geq 0$ and a vector $\xi \in \mathbb{R}^{N+1}$, $\xi \neq 0$, such that

$$0 = \sum_{i,j=0}^{N} L_w^{(i+j)}(t_0) \xi_i \xi_j = \sum_{i,j=0}^{N} \langle w_{[i]}; w_{[j]} \rangle_{t_0} \xi_i \xi_j = \left\| \sum_{i=0}^{N} w_{[i]} \xi_i \right\|_{t_0}^2 .$$

But the uniqueness theorem for analytic functions now yields that

$$\sum_{i=0}^{N} w_{[i]} \xi_i \equiv 0$$

on $\mathcal{U}$, that is the system $\{w_{[i]}\}_{i=0}^{N}$ is linearly dependent. In particular, it follows that $L_w^{(k)}(t)$ fails strict positivity for all $t \in \mathfrak{I}$.

The converse property is verified in the same manner and the assertion is proved. □

2.3. Polynomial lemniscates

Proof (of Theorem 2). To prove the first statement we fix $\mathfrak{I}_j$ to be a principal regular interval of P and let $w \equiv 1$. In our previous notations

$$L_w(t) = |E_P(t)| = \|w\|_t^2$$

Since $\Phi_P(t) = \ln|E_P(t)| - \frac{t}{n}$, we can examine only the latter logarithm. Then convexity of $\ln|E_P(t)|$ immediately follows from (22) and Cauchy's inequality:

$$L''_w(t)L_w(t) - L'^2_w(t) = \|w_{[1]}\|_t^2\|w\|_t^2 - \langle w; w_{[1]}\rangle_t^2 \geq 0, \qquad t \in \mathfrak{I}_j.$$

Let now suppose that for some $\tau \in \mathfrak{I}_j$ the equality $L''_w(\tau)L_w(\tau) - L'^2_w(\tau) = 0$ holds. Then $w_{[1]}(z) = cw(z)$, $z \in E_P(\tau)$, for a constant $c \in \mathbb{R}$. Then the uniqueness theorem for analytic functions yields that the last identity holds everywhere in $\mathcal{U}(\mathfrak{I}_j)$ and it follows from (5) that

$$2gw' + g'w = cw$$

and applying $w \equiv 1$, we get

$$c = g'(z) \equiv \left(\frac{P(z)}{P'(z)}\right)'.$$

But this yields $P(z) = c(z-a)P'(z)$, and, consequently (since P is a monic polynomial), $P(z) = (z-a)^n$ with $c = 1/n$. Thus, P must be a trivial polynomial and the first statement of the theorem is proved.

Continuity of $|E_t(P)|$ can be established as follows. We observe that by virtue of (11) the following relation holds

$$|E_P(t)| = e^{\frac{t}{n}}|E_{P_t}(0)| \tag{36}$$

for all $t \in (-\infty; +\infty]$. On the other hand, we can apply a result of Eremenko-Hayman [14, Lemma 4] which states that *the lemniscate length* $|E_P(0)|$ *is a continuous function of the coefficients of* P. Since the coefficients of P_t (see the explicit expression in (10)) are continuous functions of $t \in (-\infty; +\infty]$, the required property follows now from (36).

It remains to prove (13). Here we have $t \in (T_{\nu-1}; +\infty)$ where $T_{\nu-1}$ is the largest finite critical value of $\ln|P|$. Again, applying the Eremenko-Hayman lemma and (11) we notice that $\lim_{t\to+\infty} P_t(z) = z^n$ whence

$$\lim_{t\to+\infty} |E_t(P)|e^{-t/n} = |E_0(z^n)| = 2\pi.$$

and the theorem is proved. □

3. Applications

3.1. $\mathcal{D}$-functions

Some problems being initially posed by Piranian in [26] and dealing with monotonicity and convexity of the length function for $Q_n(z) = z^n - 1$ were studied in papers [7], [9]. Below we obtain explicit formulae for the length-functions $|E_f(t)|$ for a special class analytic functions f which include Q_n as a partial case. Our method involves representation (7) to reduce the problem to a certain hypergeometric differential equation. We demonstrate it by the following example.

Let w be an analytic function which satisfies the relation

$$(\alpha w + \beta w_{[1]}) f^{\nu} = \gamma w + \delta w_{[1]}, \tag{37}$$

Here we write as above $w_{[1]} = G_f(w) = 2gw' + g'w$. We exclude the trivial case by assuming that

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0. \tag{38}$$

Let $(\mathcal{U}, f, \mathfrak{I})$ be a regular lemniscate region. To ensure existence we can suppose that $\mathcal{Z}(f) \neq \emptyset$. Then for all $t \in \mathfrak{I}$ we have from (37)

$$e^{2\nu t} |\alpha w(z) + \beta w_{[1]}(z)|^2 = |\gamma w(z) + \delta w_{[1]}(z)|^2$$

that after integration over $E_{\mathcal{U},f}(t)$ and using (22) yields

$$(\beta^2 e^{2\nu t} - \delta^2) L''_w(t) + 2(\alpha\beta e^{2\nu t} - \gamma\delta) L'_w(t) + (\alpha^2 e^{2\nu t} - \gamma^2) L_w(t) = 0 \tag{39}$$

with $L_w(t) := \|w\|_t^2$.

Let now choose $w \equiv 1$, such that $L_w(t) = |E_{\mathcal{U},f}(t)|$ becomes the length function. Then (37) can be rewritten as

$$f^{\nu} = \frac{\gamma + \delta g'(z)}{\alpha + \beta g'(z)}.$$

Let $z_1 \in \mathcal{Z}(f)$. We can assume also that z_1 is a simple zero of $f(z)$, since the arguments similar to that given below show that the general case also leads us to the same form of the main equation (40). Then it follows from the definition of g that $g'(z_1) = 1$ and consequently we have from (37): $\gamma + \delta = 0$, whence $\gamma = -\delta$. Moreover, it follows from (38) that $\delta \neq 0$ and changing the notations $a = \alpha/\delta$ and $b = \beta/\delta$ we arrive at

$$g'(z) = \frac{af^{\nu} + 1}{1 - bf^{\nu}}.$$

Taking into account that

$$d(\ln(f/f')) = \frac{dg}{g} = \frac{af^{\nu} + 1}{1 - bf^{\nu}} \cdot \frac{df}{f} = \frac{af^{\nu} + 1}{1 - bf^{\nu}} \cdot \frac{df^{\nu}}{\nu f^{\nu}}$$

Table 1. Elementary $\mathcal{D}$-functions

$\varphi(z)$	k	ν	$p=\frac{k+1}{2\nu}$	C
$1-z^n$	$-\frac{1}{n}$	1	$\frac{n-1}{2n}$	$-n$
$\sin z$	0	2	$\frac{1}{4}$	1
$\tanh z$	1	2	$\frac{1}{2}$	1
$e^{-z}+1$	1	1	$\frac{1}{2}$	1

we obtain after integration and a suitable changing the variables $\varphi(z)=c_1 f(c_2 z)$ that

$$\varphi' = C(1-\varphi^\nu)^{\frac{k+1}{\nu}}, \tag{40}$$

with $k=a/b$.

To analyze the last equation we suppose that ν is a positive integer and $k+1\le\nu$. Then the set of solutions to (40) is still large and such elementary functions as $\tanh z$, $\sin z$, z^n-1, $e^{-z}+1$ satisfy these conditions (see Table 1).

First, we notice that under our assumptions the integral

$$F(z)=\int_0^z \frac{d\zeta}{(1-\zeta^\nu)^{\frac{k+1}{\nu}}},$$

where the principal branch of the root is chosen, defines a univalent function $F(z)$ in the unit disk $\mathbb{D}=\{z : |z|<1\}$ since the real part of the derivative

$$\operatorname{Re} F'(z)=\frac{\operatorname{Re}(1-z^\nu)^{\frac{k+1}{\nu}}}{|1-z^\nu|^{\frac{2(k+1)}{\nu}}}>0, \qquad z\in\mathbb{D}$$

is positive and we can apply the Noshiro-Warschawski theorem [8, p. 47]. Thus, for any $C\neq 0$ the function $\varphi(z)$ given by

$$F(\varphi(z))=Cz$$

is analytic (and univalent) in $S:=\frac{1}{C}F(\mathbb{D})$, $\varphi : S\to\mathbb{D}$ and satisfies (40) there. We call such a function φ a *$\mathcal{D}$-function*.

An important property of $\mathcal{D}$-functions is that their critical values have the same magnitude: $\varphi'(\zeta)=0 \Rightarrow |\varphi(\zeta)|=1$. Clearly, $(S^*,\varphi,(-\infty,0))$ is a regular lemniscate region of the function φ, where $S^*=S\setminus\{0\}$.

Another representation of F can be easily found by using the Gauss hypergeometric function

$$F(\zeta)=\zeta\ {}_2F_1\left(\frac{1+k}{\nu},\frac{1}{\nu};\frac{1+\nu}{\nu},\zeta^\nu\right), \qquad \zeta\in\mathbb{D}.$$

Theorem 3. *Let $\nu \in \mathbb{Z}^+$ and $k \le \nu - 1$. Then in the preceding notations, the following formula holds*

$$|E_{S^*,\varphi}(t)| = \frac{2\pi e^t}{|C|}\, {}_2F_1(p, p; 1; e^{2\nu t}), \tag{41}$$

where $p = (k+1)/2\nu$ and ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function.

Proof. Let $\varphi(z)$ a $\mathcal{D}$-function with parameters ν, k, and C. Then

$$g'_\varphi(z) \equiv \frac{\varphi(z)}{\varphi'(z)} = \frac{1 + k\varphi^\nu}{1 - \varphi^\nu},$$

and after comparing with (37) we have $\alpha = k$, $\beta = 1$, $\gamma = -\delta = -1$. We notice that in our previous notations the function $|E_{S^*,\varphi}(t)| = L_w(t)$ is a solution of (39) where $w \equiv 1$. Hence, letting $t = \ln\tau$ and

$$L(\tau) := |E_{S^*,\varphi}(\ln\tau)|$$

one can readily see from (39) that

$$\tau^2(\tau^{2\nu} - 1)L''(\tau) + [1 + (2k+1)\tau^{2\nu}]\tau L'(\tau) + (k^2\tau^{2\nu} - 1)L(\tau) = 0. \tag{42}$$

Further, the change of variables $x = \tau^{2\nu}$ and $L(\tau) = \tau y(\tau^{2\nu})$ reduces (42) to the hypergeometric canonic form

$$x(1-x)y''(x) + \left[1 - x\left(1 + \frac{k+1}{\nu}\right)\right] y'(x) - \left(\frac{k+1}{2\nu}\right)^2 y(x) = 0. \tag{43}$$

The general solution to (43) in [0; 1) can be written as follows

$$y(\tau) = {}_2F_1(p, p; 1; \tau)\,\lambda + {}_2F_1(p, p; 2p; 1-\tau)\,\mu,$$

where $\lambda, \mu \in \mathbb{R}$ (see [10, § 2.3.1]).

Taking into account that $z = 0 \in S$ is a simple zero of φ we conclude that the length function $|E_{S^*,\varphi}(t)| = O(e^t)$ as $t \to -\infty$. Thus, $y(\tau)$ is a bounded function near $\tau = +0$. Since

$$\lim_{\tau\to+0} {}_2F_1(p, p; 2p; 1-\tau) = \infty$$

(see [10, § 2.1.3])) we have $\mu = 0$ and after substitution of the old notations we arrive at

$$|E_{S^*,\varphi}(t)| = \lambda e^t\, {}_2F_1(p, p; 1; e^{2\nu t}).$$

The precise form of λ can now be found by the asymptotic behavior

$$|E_{S^*,\varphi}(t)| = \frac{2\pi e^t}{|\varphi'(0)|}, \qquad t \to -\infty,$$

On the other hand, by the definition of $\mathcal{D}$-function $|\varphi'(0)| = |C|$ and the theorem is proved. □

Remark 4. In the case $t > 0$, the last theorem is still meaningful provided that the level sets $\{|\varphi(z)| = e^t\}$ are compact or *periodic* curves (the last, e.g., corresponds to $\varphi = \sin z$). These cases are treated in [21].

3.2. The structure of $\sigma^{P,\mathfrak{I}}$

Here we describe an explicit structure of the measure function (for monic polynomials) $\sigma^{P,\mathfrak{I}_j}(x)$ when $j = \nu$, i.e. the interval $(T_{\nu-1}, +\infty)$ is free of critical values of $\ln|P|$. The latter means that the corresponding lemniscates $E_P(t)$, $t \in \mathfrak{I}_\nu$, are single-component closed curves.

Theorem 4. *Let P be a monic polynomial of degree n and*

$$T(P) := T_\nu = \max_{P'(\zeta_k)=0} \ln|P(\zeta_k)|$$

is the largest singular value. Then for all $t \geq T(P)$ the following representation holds

$$|E_P(t)| = 2\pi e^{t/n}\left(1 + \sum_{k=2}^{+\infty} |c_{-k}|^2 e^{-2kt/n}\right), \tag{44}$$

and

$$\sum_{k\geq 2} |c_{-k}|^2 e^{-2kT(P)/n} < +\infty. \tag{45}$$

Here c_k is the kth Laurent coefficient of $\sqrt{\varphi'(\zeta)}$ near the infinity, where $P(\varphi(\zeta)) = \zeta^n$.

Proof. Let $t > T(P)$ be chosen arbitrary. Then $E_P(t)$ is a simple Jordan curve which is the boundary of a simply-connected domain

$$D_t = \{z \in \mathbb{C} : |P(z)| < e^t\}.$$

Let $D_t^* = \overline{\mathbb{C}} \setminus \overline{D_t}$. Then $P(z)$ maps $D^*(t)$ onto $\mathbb{U}_t := \{\zeta \in \overline{\mathbb{C}} : |\zeta| > e^t\}$. Since $P'(z) \neq 0$ in $\mathbb{U}_t$ the analytic function $F(z) = P^{1/n}(z)$, $F(z) \sim z$ as $z \to \infty$, is well defined in $D^*(t)$. Moreover, $F(z)$ is univalent in $D^*(t)$ for all $t \geq T(P)$. We denote by $\varphi(\zeta)$ the inverse function which is also a univalent function and observe that

$$\varphi : \mathbb{U}_{T(P)/n} \to D^*(T(P)); \qquad \varphi'(\zeta) \neq 0, \quad \zeta \in \mathbb{U}_{T(P)/n}. \tag{46}$$

We have $\varphi(\zeta) \sim \zeta$ as $\zeta \to \infty$ and letting $P(z) = z^n + a_1 z^{n-1} \ldots + a_n$ we obtain

$$\varphi'(\zeta) = 1 - \frac{(n-1)a_1^2 - 2na_2}{2n^2}\frac{1}{\zeta^2} + \cdots, \qquad \zeta \to \infty. \tag{47}$$

It follows that $\sqrt{\varphi'}$ is a well-defined analytic function in $\overline{\mathbb{U}}_{T(P)/n}$ (for sake of completeness let $\sqrt{1} = 1$). Thus, it can be expanded in the Laurent series

$$\sqrt{\varphi'(\zeta)} = 1 + \sum_{k=2}^{+\infty} \frac{c_{-k}}{\zeta^k}, \qquad |\zeta| > e^{T(P)/n}. \tag{48}$$

Next, we observe that for $t \geq T(P)$, the curve $E_P(t)$ is homeomorphic to a circle and can be naturally parameterized by

$$E_P(t) = \{\varphi(\theta) : \quad \theta \in e^{t/n}\mathbb{T}\},$$

where $\mathbb{T} = \partial\mathbb{D}$ is the unit circle, and we have for the length function

$$\begin{aligned} |E_P(t)| &= \int_{e^{t/n}\mathbb{T}} |\varphi'(\zeta)|\,|d\zeta| = \int_{e^{t/n}\mathbb{T}} |\sqrt{\varphi'(\zeta)}|^2\,|d\zeta| = \\ &= \int_{e^{t/n}\mathbb{T}} \sum_{k=-\infty}^{+\infty} |c_k|^2|\zeta|^{2k}\,|d\zeta| = 2\pi e^{t/n}\left(1+\sum_{k=2}^{+\infty}|c_{-k}|^2 e^{-2kt/n}\right) \end{aligned} \tag{49}$$

where convergence of the corresponding series for $t > T(P)$ follows from that one in (48).

To establish (45) we note that it follows from (49) that $|E_P(t)|$ is a decreasing function in $(T(P); +\infty)$ and continuous in $[T(P), +\infty)$ (see Theorem 2). Thus, the right-hand side of (49) is at most $|E_P(T(P))|$ for all $t > T(P)$. Then it follows from positivity of the terms of the corresponding series and the mentioned continuity that (49) is still true for $t = T(P)$, which in turn yields (45) and completes the proof. □

Corollary 9. *In the notations of Theorem 4 we have the following lower estimate*

$$e^{4t/n}\left(\frac{e^{-t/n}}{2\pi}|E_P(t)| - 1\right) \geq \left|\frac{(n-1)a_1^2 - 2na_2}{4n^2}\right|^2, \qquad t \geq T(P) \tag{50}$$

where $P(z) = z^n + a_1 z^{n-1} \ldots + a_n$ and the estimate is sharp.

Proof. The estimate easily follows from (47). On the other hand, the same formula shows that the left-hand side of (50), denote it by $h(t)$, is a decreasing function as $t \geq T(P)$ and

$$\lim_{t\to+\infty} h(t) = |c_{-2}|^2 = \left|\frac{(n-1)a_1^2 - 2na_2}{4n^2}\right|^2$$

which proves a sharp character of (50). □

It is helpful also to notice that the expression in the right-hand side of (50) has the form

$$\frac{(n-1)a_1^2 - 2na_2}{4n^2} = \frac{1}{4n^2}\sum_{1\leq i<j\leq n}(z_i - z_j)^2$$

where $P(z) = \prod_{j=1}^n (z - z_j)$.

We recall (see [30, p. 145]) that a function $f(t)$ is *completely monotonic* in (a, b) if it has non-negative derivatives of all orders there:

$$(-1)^k f^{(k)}(t) \geq 0. \tag{51}$$

A function $f(t)$ is said to be completely monotonic in $[a, b)$ if it is continuous there and satisfies (51) in (a, b).

Corollary 10. $|E_P(t)|e^{-t/n}$ *is completely monotonic in* $[T(P), +\infty)$.

Finally, we briefly discuss the mentioned in Corollary 2 analytic continuation property. In the post critical case $t > T(P)$ this fact can be established directly. Indeed, let

$$F(z) = 2\pi e^z \left(1 + \sum_{k=2}^{+\infty} |c_{-k}|^2 e^{-2kz}\right), \qquad \operatorname{Re} z \geq \frac{T(P)}{n}, \tag{52}$$

then it follows from (45) that $F(\zeta)$ is a single-valued analytic function and

$$|E_P(t)| = F\left(\frac{t}{n}\right), \qquad t \in \mathbb{R} \tag{53}$$

is a desirable continuation.

Moreover, we note that $F(z)$ is a $2\pi i$-periodic function. Let

$$\lambda(\zeta) = F(\ln \zeta) = 2\pi\zeta \left(1 + \sum_{k=2}^{+\infty} \frac{|c_{-k}|^2}{\zeta^{2k}}\right), \qquad |\zeta| > e^{\frac{T(P)}{n}}$$

Then the last formula shows that $\lambda(\zeta)$ is an *odd* analytic function and

$$|E_P(t)| = \lambda(e^{t/n}), \quad \forall t \geq T(P). \tag{54}$$

Acknowledgements. The authors are grateful to Alexander Eremenko, Björn Gustafsson, Henrik Shahgholian, Harold Shapiro and Serguei Shimorin for their constructive comments. We are also indebted to Klaus Steffen and the referee for careful reading and suggestions which led to a considerable improvement of the paper.

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