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FINITENESS OF THE NUMBER OF ENDS OF MINIMAL SUBMANIFOLDS IN EUCLIDEAN SPACE

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We prove a version of the well-known Denjoy-Ahlfors theorem about the number of asymptotic values of an entire function for properly immersed minimal surfaces of arbitrary codimension in $\mathbf{R}^N$. The finiteness of the number of ends is proved for minimal submanifolds with finite projective volume. We show, as a corollary, that a minimal surface of codimension n meeting any n-plane passing through the origin in at most k points has no more $c(n, N)k$ ends.

Let $x : M \to \mathbf{R}^n$ be a proper minimal immersion of a p-dimensional orientiable manifold M. Then it is well-known that M is necessarily noncompact. The simplest topological invariant of such manifolds is *the number of infinite points* (*or ends*) of M, i.e. the smallest integer $\ell(M)$ satisfying the following property: for every compact set $F \subset M$ the number of the different components with noncompact closure of $M \setminus F$ is less or equal to $\ell(M)$.

We say that a manifold M (or the properly immersed surface $\mathcal{M} = (M, x)$) is *manifold* (*surface* – respectively) *with finitely many ends* if $\ell(M) < +\infty$.

These definitions agree with the usual ones for Riemannian surfaces of finite type (see Example 1 below) and are related to the noncompactness of the manifold.

In this paper we obtain some upper bounds for $\ell(M)$ in terms of the *projective volume* $V_p(\mathcal{M})$ of M and certain integral-geometric characteristics related to the geometry in the large of minimal surfaces.

If $\dim M = 2$ and M has finite total curvature $K(M)$, R.Osserman [7] (see also [10]) proved that M is conformally equivalent to a compact Riemann surface that has been punctured in a finite number of points $\{m_1, m_2, \ldots, m_k\}$. In this case $\ell(M)$ is equal to k. We observe, however, that the quantity $K(M)$ itself does not represent any information about $\ell(M)$. Furthermore, there exist minimal surfaces of finite topological type with $K(M) = -\infty$. For a detailled discussion of these questions we refer to [4], [5].

The projective volume is one of the main tools in uniformization theory and potential theory. Using the special technique of estimating extremal lengths in terms of a projective volume, V.M.Miklyukov and the author in [6] showed that a minimal surface $\mathcal{M}$ in $\mathbf{R}^3$ has parabolic conformal type provided that the generic number of points which $\mathcal{M}$ has in common with a line L passing through a fixed point (possibly the infinitely far one) in $\mathbf{R}^3$ is uniformly bounded on L. In particular, an upper bound for the projective volume of the such surfaces was established.

In part 2 we prove that $\ell(M)$ is bounded by $c(p,n)V_p(\mathcal{M})$. We consider Theorem 2 as an extension of the Denjoy-Ahlfors theorem about the number of asymptotic values (see the beautiful review of A.Baernstein [2]) to minimal submanifolds. As a corollary we obtain in part 3 that a p-dimensional properly immersed minimal surface meeting any $(n-p)$-plane, passing through origin, in at most k points has no more than $c(p,n)k$ ends. In particular, if a minimal hypersurface $\mathcal{M}$ is starlike with respect to some point, then the number $\ell(M)$ is less than a constant depending only on $\dim\mathcal{M}$.

The results proved in this paper allow also to infer a parabolic conformal type for properly immersed minimal submanifolds of arbitrary codimension in the same way as in [6]. We wish to mention also the paper [9] devoted to the study of surfaces of hyperbolic type and [8] where results close to ours has been obtained.

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1. Some properties of the projective volume

Let $a \in \mathbf{R}^n$ and $\mathcal{P}_a$ be the group consisting of all conformal transformations preserving the set $\{a, \infty\}$, i.e $\mathcal{P}_a$ is generated by the inversions: $x \rightarrow \lambda\ (x\ -\ a\ )\ |\ x\ -\ a\ |^{-2}$ and the homotheties: $x \rightarrow \lambda(x-a)$, where λ is a positive factor.

Let $\mathcal{M}$ be a p-dimensional surface in $\mathbf{R}^n$ and $B_a(R)$ be a ball $\{x \in \mathrm{R}^n : |x-a| < R\}$. We denote by $M_a(R)$ the part of the surface $\mathcal{M}$ inside $B_a(R)$ and abbreviate $x_a(m) = x(m) - a$, i.e. $x_a(m)$ is a radius-vector of $\mathcal{M}$ associated with $a \in \mathrm{R}^n$. For given $a \in \mathbf{R}^n \setminus x(M)$ we define the following metric characteristic of $\mathcal{M}$:

$$V_p(\mathcal{M}, a) = \limsup_{R\to\infty} \frac{1}{\ln R} \int\limits_{M_a(R)} \frac{1}{|x_a(m)|^p}. \tag{1}$$

It is easy to see that $V_p(\mathcal{M}, a)$ is invariant under the action of the group $\mathcal{P}_a$. We call $V_p(\mathcal{M}, a)$ the ***projective volume*** of $\mathcal{M}$.

Let $y^{\perp}(m)$ be the projection of y on the normal space to the surface $\mathcal{M}$ at a point m. Then we let

$$Q_p(\mathcal{M},a) = \int_M \frac{|x_a^{\perp}(m)|^2}{|x_a(m)|^{p+2}} = \int_M \frac{|(x(m)-a)^{\perp}|^2}{|x(m)-a|^{p+2}},$$

and $Q_p(\mathcal{M},a) = +\infty$, if the last integral is divergent. For $a \in x(M)$ we set

$$Q_p(\mathcal{M},a) = \lim_{\epsilon\to 0} \int_{|x_a(m)|>\epsilon} \frac{|x_a^{\perp}(m)|^2}{|x_a(m)|^{p+2}}.$$

Theorem 1. *Let $\mathcal{M}$ be a properly immersed p-dimensional minimal surface in $\mathbf{R}^n$ with compact boundary Σ. Then the value $V_p(\mathcal{M},a)$ does not depend on the choice of $a \in \mathbf{R}^n \setminus x(M)$. Moreover, the upper limit in (1) can be replaced by a limit and*

$$pQ_p(\mathcal{M},a) = V_p(\mathcal{M},a) + c(\Sigma;a), \tag{2}$$

where $c(\Sigma;a)$ is a finite constant such that $c(\varnothing;a) = 0$.

Proof. Let us assume $a \notin x(M)$. Denote $h = \operatorname{dist}(a, x(M))$, $r = \max_{m\in\Sigma} |x(m)-a|$ ($r=0$, if $\Sigma = \varnothing$) and $\varrho = \max\{h;r\}$. It is obvious from the properness of the immersion, that $\varrho > 0$. Letting $f(m) = |x(m)-a|$ we have

$$\nabla f(m) = \frac{x_a^{\mathsf{T}}(m)}{|x_a(m)|}$$

where $(\ )^{\mathsf{T}}$ is the tangent part of the corresponding vector and hence

$$\operatorname{div}\frac{x_a^{\mathsf{T}}(m)}{|x_a(m)|^p} = \frac{1}{|x_a(m)|^p}\operatorname{div}(x_a^{\mathsf{T}}(m)) - \frac{p}{|x_a(m)|^{p+1}}\langle \nabla f, x_a^{\mathsf{T}}(m)\rangle$$

$$= \frac{p(|x_a(m)|^2 - |x_a^{\mathsf{T}}(m)|^2)}{|x_a(m)|^{p+2}} = \frac{p|x_a^{\perp}(m)|^2}{|x_a(m)|^{p+2}}. \tag{3}$$

Applying Stokes' formula in the last identity over $M_a(t,R) \equiv M_a(R) \setminus \overline{M_a(t)}$ for $\varrho < t < R$ we have

$$\frac{1}{R^p} \int\limits_{\partial M_a(R)} \langle x_a^{\top}(m), \nu\rangle - \frac{1}{t^p} \int\limits_{\partial M_a(t)} \langle x_a^{\top}(m), \nu\rangle = p \int\limits_{M_a(t,R)} \frac{|x_a^{\perp}(m)|^2}{|x_a(m)|^{p+2}},$$

where ν is the unit outward normal to the t-level set $\partial M_a(t)$ of the function $f(m)$. It is easy to see that for any regular value $t > \varrho$ of f the normal ν is represented on $\partial M_a(t)$ by

$$\nu(m) = \frac{x_a^{\top}(m)}{|x_a^{\top}(m)|},$$

and thus the last integral expression can be rewritten in the form

$$\frac{J(R)}{R^{p-1}} - \frac{J(t)}{t^{p-1}} = p \int\limits_{M_a(t,R)} \frac{|x_a^{\perp}(m)|^2}{|x_a(m)|^{p+2}}, \tag{4}$$

where $J(t) = t^{-1} \int_{\partial M_a(t)} |x_a^{\top}(m)|$. This relation is a well-known monotonicity formula for minimal surfaces and (4) yields the increasing monotonicity of $J(t)t^{1-p}$.

On the other hand using the Kronrod-Federer formula ([3], Theorem 3.2.22), we obtain for $R > R_1 > \varrho$,

$$\int\limits_{M_a(R_1,R)} \frac{1}{|x_a(m)|^p} = \int\limits_{M_a(R_1,R)} \frac{|x_a^{\top}(m)|^2}{|x_a(m)|^{p+2}} + \int\limits_{M_a(R_1,R)} \frac{|x_a^{\perp}(m)|^2}{|x_a(m)|^{p+2}}$$

$$= \int_{R_1}^{R} \frac{dt}{t^{p+2}} \int\limits_{\partial M_a(t)} \frac{|x_a^{\top}(m)|^2}{|\nabla f|} + \int\limits_{M_a(R_1,R)} \frac{|x_a^{\perp}(m)|^2}{|x_a(m)|^{p+2}}$$

$$= \int_{R_1}^{R} \frac{J(t)}{t^{p-1}} \frac{dt}{t} + \int\limits_{M_a(R_1,R)} \frac{|x_a^{\perp}(m)|^2}{|x_a(m)|^{p+2}}$$

$$= \int_{R_1}^{R} \frac{J(t)}{t^{p-1}} \frac{dt}{t} + \frac{1}{p}\left(\frac{J(R)}{R^{p-1}} - \frac{J(R_1)}{R_1^{p-1}}\right). \tag{5}$$

Now we notice that the increasing monotonicity of $J(t)t^{1-p}$ yields immediately

$$\lim_{R\to+\infty} \frac{J(R)}{R^{p-1}} = V_p(\mathcal{M}, a), \tag{6}$$

and consequently,

$$\frac{J(R)}{R^{p-1}} \le V_p(\mathcal{M}, a), \quad \text{for} \quad R > \varrho. \tag{7}$$

Then we deduce from (6) and (5) that the upper limit in (1) can be replaced on a limit.

Now we show that really $V_p(\mathcal{M}, a)$ does not depend on a. Let b be a point in $\mathbf{R}^n \setminus x(M)$ and $\delta = |b-a|$. Then for every $0 < \varepsilon < 1$ there exist $R(\varepsilon) > \delta$ and $c(\varepsilon) < \infty$ (both independent of R) such that for $R > R(\varepsilon)$ one has

$$\left| \int\limits_{M_b(R)} \frac{1}{|x-b|^p} - \int\limits_{M_b(R)} \frac{1}{|x-a|^p} \right| \le \int\limits_{M_b(R(\varepsilon))} \left| \frac{1}{|x-a|^p} - \frac{1}{|x-b|^p} \right|$$

$$+ p\delta \int\limits_{M_b(R(\varepsilon),R)} \frac{|x-a|^{p-1} + |x-b|^{p-1}}{|x-b|^p|x-a|^p} \le c(\varepsilon) + \varepsilon \int\limits_{M_b(R)} \frac{1}{|x-b|^p}, \tag{8}$$

(choosing $R(\varepsilon) >> 1$ such that $\frac{p\delta}{R(\varepsilon)-\delta}\left[\left(\frac{R(\varepsilon)}{R(\varepsilon)-\delta}\right)^{p-1} + 1\right] \le \varepsilon$).

Moreover, from the obvious inclusions

$$M_a(R-\delta) \subset M_b(R) \subset M_a(R+\delta),$$

for $R > \delta$, we have

$$\int\limits_{M_a(R-\delta)} \frac{1}{|x-b|^p} \le \int\limits_{M_b(R)} \frac{1}{|x-b|^p} \le \int\limits_{M_a(R+\delta)} \frac{1}{|x-b|^p}.$$

From this and (8), it then follows that

$$\begin{aligned} \frac{1}{1+\varepsilon} \int\limits_{M_a(R-\delta)} \frac{1}{|x-a|^p} - \frac{c(\varepsilon)}{1+\varepsilon} &\le \int\limits_{M_b(R)} \frac{1}{|x-b|^p} \\ &\le \frac{1}{1-\varepsilon} \int\limits_{M_a(R+\delta)} \frac{1}{|x-a|^p} + \frac{c(\varepsilon)}{1-\varepsilon} \end{aligned}$$

and dividing by $\ln R$ one infers

$$\frac{1}{1+\varepsilon}V_p(\mathcal{M},a) \leq V_p(\mathcal{M},b) \leq \frac{1}{1-\varepsilon}V_p(\mathcal{M},a)$$

letting $R \to \infty$. In view of arbitrariness of ε this implies $V_p(\mathcal{M},a) = V_p(\mathcal{M},b)$ whether both quantities are finite or not.

Now integrating (3) over $M_a(R)$, we obtain for $R > \varrho$

$$\frac{1}{R^p} \int_{\partial M_a(R)} \langle x_a^{\top}(m), \nu\rangle - \int_{\Sigma} \frac{\langle x_a^{\top}(m), \nu\rangle}{|x_a(m)|^p} = p \int_{M_a(R)} \frac{|x_a^{\perp}(m)|^2}{|x_a(m)|^{p+2}}.$$

Let us denote by $c(\Sigma; a)$ the second integral in the above equation. Then we have

$$\frac{J(R)}{R^{p-1}} - c(\Sigma; a) = p \int_{M_a(R)} \frac{|x_a^{\perp}(m)|^2}{|x_a(m)|^{p+2}}.$$

Leting $R \to \infty$ and using the equality (6), we complete the proof of Theorem 1.

We denote by $a\#\mathcal{M}$ the multiplicity of the immersion $x : M \to \mathbf{R}^n$ at the point $a \in \mathbf{R}^n$, i.e. the cardinal number of the preimage $x^{-1}(a \cap x(M))$.

The next property of $V_p(\mathcal{M},a)$ and $Q_p(\mathcal{M},a)$ shows that these quantities are *conformal invariants* of minimal surfaces in the sense that $V_p(g \circ \mathcal{M}; a) = V_p(\mathcal{M}; 0)$ for all $g \in \mathcal{P}_a$, and $a \in \mathbf{R}^n \setminus x(M)$.

Corollary 1. *Let $\mathcal{M} \subset \mathbf{R}^n$ be a properly immersed minimal surface without boundary,* $\dim \mathcal{M} = p$*. Then both values $Q_p(\mathcal{M},a)$ and $V_p(\mathcal{M},a)$ do not depend on choice of $a \in \mathbf{R}^n \setminus x(M)$ and for all $a \in \mathbf{R}^n$ we have*

$$\frac{1}{p}\, V_p(\mathcal{M},a) = Q_p(\mathcal{M},a) - \omega_p(a\#\mathcal{M}),$$

where ω_p is the $(p-1)$-dimensional Hausdorff measure of unit sphere S^{p-1}.

Proof. We observe that the first assertion of Corollary 1 follows immediately from $\Sigma = \varnothing$ and (2).

Let us now consider $a \in x(M)$ so that $q = a\#\mathcal{M}$ is a positive integer. Reasoning similarly as above we get that $J(t)t^{1-p}$ is a positive monotonic function for $t \to +0$. Consequently, there exists

$$\mu = \lim_{t\to+0} \frac{J(t)}{t^{p-1}}.$$

We consider any preimage $m_k \in x^{-1}(a)$. Let $\mathcal{O}_k(t)$ be an open component of $M_a(t)$ which contains m_k. It is clear that for sufficiently small $t > 0$ the sets $\mathcal{O}_k(t)$ are nonintersecting for all $k \leq q$. Then from (4) we have

$$\frac{J(R)}{R^{p-1}} - \mu = p \int\limits_{M_a(R)} \frac{|x_a(m)^{\perp}|^2}{|x_a(m)|^{p+2}}. \tag{9}$$

But in virtue of the regularity of the immersion $x(m)$,

$$\lim_{t\to+0} \frac{1}{t} \sup_{m\in\mathcal{O}_k(t)} |x_a^{\perp}(m)| \quad = 0.$$

It follows that

$$\lim_{t\to+0} \frac{1}{t^{p-1}} \int\limits_{\partial\mathcal{O}_k(t)} \frac{|x_a^{\top}(m)|}{|x_a(m)|} = \lim_{t\to+0} \frac{\operatorname{meas}_{p-1}(\partial\mathcal{O}_k(t))}{t^{p-1}} = \omega_p, \tag{10}$$

and taking into consideration that for small $t > 0$

$$\partial M_a(t) = \bigcup_{k=1}^{q} \partial\mathcal{O}_k(t),$$

we obtain $\mu = q\,\omega_p$.

Repeating the above arguments we conclude

$$\lim_{R\to\infty} \frac{1}{\ln R} \int\limits_{M_a(R)} \frac{1}{|x_a(m)|^p} = \lim_{R\to+\infty} \frac{J(R)}{R^{p-1}} \quad ,$$

and by (9) and (10) Corollary 1 is proved.

From now on we write $Q_p(\mathcal{M})$ and $V_p(\mathcal{M})$ instead of $Q(\mathcal{M}, a)$ and $V_p(\mathcal{M}, a)$ respectively, if $a \notin x(\mathcal{M})$.

Example 1. Let $\mathcal{M}_g$ be a compact orientiable Riemannian surface $\mathcal{M}_g$ of genus $g \geq 0$, and $m_1, m_2, \dots m_l \in \mathcal{M}_g$. Let ζ be a holomorphic 1-form on $\mathcal{M}_g$ and $h : \mathcal{M}_g \to \mathbb{C} \cup \{\infty\}$ a meromorphic function. Then due to [7] the vector valued 1-form

$$\Phi = (\Phi_1, \Phi_2, \Phi_3)^t = \Big((1 - h^2)\zeta;\ i(1 + h^2)\zeta;\ 2h\zeta\Big)^t$$

gives a conformal minimal immersion

$$X(m) = \mathrm{Re} \int_{m_0}^{m} \Phi$$

which is well-defined on $\mathcal{M}_g^\star = \mathcal{M}_g \setminus \{m_1, m_2, \dots, m_l\}$ and regular, provided

1. *No component of Φ has a real period on $\mathcal{M}_g$;*
2. *The poles $\{m_1, m_2, \dots, m_l\}$ of h coincide with zeros of ζ and the order of a pole m_k of h is precisely the order of the corresponding zero of ζ.*

It is well-known in the case of finite total curvature that the asymptotic behaviour of $X(m)$ in the neighbourhood of m_k is either of flat or catenoid type [5]. In both cases the quantity $Q_p(\mathcal{M})$ and, consequently, $V_p(\mathcal{M})$ can be calculated directly and we have

$$V_p(\mathcal{M}) \equiv 2Q_p(\mathcal{M}) = 2\pi l\,.$$

We observe that *the characteristic $Q(\mathcal{M})$ does not depend on the genus g of $\mathcal{M}_g$, and describes only the noncompactness nature of $\mathcal{M}_g^\star$.*

Remark 1. It would be interesting to know in analogy with the case of finite Gaussian curvature above, *whether the set of possible* *values of the quantity* $Q(\mathcal{M})$ *is discrete.* It follows from the above example that this is true for two-dimensional minimal surfaces of finite topology.

2. The estimate for the number of ends of minimal submanifolds

In this section we give a geometric application of the above invariants.

Theorem 2. *Let* $\mathcal{M}$ *be a properly immersed p-dimensional minimal surface in* $\mathbb{R}^n$ *with compact boundary* Σ*, having finite projective volume* $V_p(\mathcal{M})$*. Then* $\mathcal{M}$ *is a surface with finitely many ends and*

$$\ell(M) \leq \frac{2^p}{\omega_p} V_p(\mathcal{M}).$$

The proof of the theorem is based on the next auxiliary assertion.

Lemma 1. *Let* $\mathcal{D}$ *be a connected p-dimensional minimal surface with boundary* $\partial\mathcal{D} \subset \partial B_0(R_1) \cup \partial B_0(R_2)$, $\quad R_2 > R_1 > 0$. *Then*

$$\text{meas}_p\mathcal{D} \geq \frac{\omega_p}{p}\left(\frac{R_2 - R_1}{2}\right)^p. \tag{11}$$

Proof of the lemma. We consider first the case when $\mathcal{D}$ is a compact minimal submanifold such that $0 \in \mathcal{D}$ and $\partial\mathcal{D} \subset \partial B_0(R)$. Let

$$A(t) = \text{meas}_p(\mathcal{D} \cap B_0(t)).$$

Then using

$$\text{div}x^\top(m) = \sum_{i=1}^n \text{div}x_i e_i^\top = \sum_{i=1}^n |e_i^\top|^2 = p$$

we shall have after integration

$$pA(t) = \int\limits_{\mathcal{D}\cap B_0(t)} \operatorname{div} x^{\top}(m) = t \int\limits_{\mathcal{D}\cap\partial B_0(t)} \frac{|x^{\top}(m)|}{|x(m)|} = t\, J(t), \tag{12}$$

that the function

$$\frac{p\, A(t)}{t^p} = \frac{J(t)}{t^{p-1}}$$

is an increasing one. Moreover,

$$\lim_{t\to+0} \frac{p\, A(t)}{t^p} = \lim_{t\to+0} \frac{J(t)}{t^{p-1}} = \omega_p \cdot (0\#\mathcal{D}),$$

and consequently, for all $t > 0$ we have

$$\frac{\operatorname{meas}_p(\mathcal{D}\cap B_0(t))}{t^p\omega_p} \geq \lim_{t\to+0} \frac{A(t)}{t^p} = \frac{1}{p}\,(0\#\mathcal{D}) \tag{13}$$

and the first case of the lemma is proved.

Let us now assume that $\partial\mathcal{D} \subset \partial B_0(R_1) \cup \partial B_0(R_2)$. Put $R = \frac{1}{2}(R_1 + R_2)$. We observe that the set $\mathcal{D}\cap\partial B_0(R)$ is not empty by virtue of the connectivity $\mathcal{D}$, and we let a be any point in $\mathcal{D}\cap\partial B_0(R)$. Then for $\mathcal{D}_1 = \mathcal{D}\cap B_0(r)$ we have

$$\partial\mathcal{D}_1 \subset \partial B_0(r)$$

for $r = \frac{1}{2}(R_2 - R_1)$. In view of (13) and the inclusion $\mathcal{D}_1 \subset \mathcal{D}$, the above inclusion implies (11) and thus the proof the lemma is concluded.

Remark 2. We note that Lemma 1 can be also obtained from the general result of W.K.Allard [1].

Proof of Theorem 2. Without loss of generality we can arrange that $0 \notin \overline{\mathcal{M}}$ and, by Theorem 1, $V_p(\mathcal{M}) = V_p(\mathcal{M}, 0)$. We fix a sufficiently large regular value $R > 0$ of $f(m) = |x(m)|$ such that $\Sigma \subset B_0(R)$.

Let $\mathcal{D}_1, \ldots, \mathcal{D}_k \ldots$ be the open components of $M \setminus \overline{M_0(R)}$. Notice that $x(\partial\mathcal{D}_k) \subset \partial B_0(R)$ and $\Delta\, f(m) \geq 0$. Then the maximum principle implies that the $\mathcal{D}_k$ are domains with noncompact closure. Moreover, it follows from the regularity of R that the number $l = l(R)$ of components $\mathcal{D}_k$ is finite, and it is nondecreasing with respect to R. Put for $t > R$,

$$J_k(t) = \frac{1}{t} \int\limits_{\partial B_0(t) \cap \mathcal{D}_k} |x^{\top}(m)|.$$

Then reasoning similarly as in the proof of Theorem 1 we arrive at the inequality

$$\sum_{k=1}^{l} J_k(t) = J(t) \;\leq\; V_p(\mathcal{M}) t^{p-1}. \tag{14}$$

On the other hand, applying Lemma 1, we have

$$\text{meas}_p\{m \in \mathcal{D}_k : \; |x(m)| < t\} \geq \frac{\omega_p (t-R)^p}{p\, 2^p},$$

and after summing over all $k \leq l$ we obtain that

$$\text{meas}_p(M_0(t) \setminus M_0(R)) \geq \frac{l\omega_p}{p 2^p}(t-R)^p.$$

Using (12) and (14) we have the sequence of inequalities:

$$\frac{l\omega_p}{p2^p}(t-R)^p \leq \text{meas}_p(M_0(t) \setminus M_0(R)) \leq \text{meas}_p(M_0(t)) =$$

$$= \frac{tJ(t)}{p} \leq \frac{V_p(\mathcal{M}) t^p}{p},$$

and after dividing by t^p and letting $t \to \infty$, we obtain

$$l(R) = l \leq \frac{V_p(\mathcal{M}) 2^p}{\omega_p}.$$

Next, from the fact that the integer-valued function $l(t)$ is nondecreasing, we conclude that it is stabilized, i.e. $l(R) \equiv \text{const}$ for sufficient large R.

Let $F \subset M$ be an arbitrary compact subset. Using again the maximum principle and the properness of the immersion $x(m)$ we obtain that the number of components of $M \setminus F$ with noncompact closure is a nondecreasing function of the compact set F. Therefore, $\ell(M) \equiv \lim_{t\to\infty} l(t)$ and the theorem is proved.

Corollary 2. *Let $\mathcal{M}$ be a properly immersed p-dimensional minimal surface without boundary having finite projective volume. Then $\mathcal{M}$ is a surface with finitely many ends and*

$$\ell(M) \leq \frac{Q_p(\mathcal{M})2^p p}{\omega_p}.$$

3. The bounded integral-geometric averages and the finiteness of the number of ends of minimal submanifolds

In this section we discuss certain sufficient conditions for the finiteness of the projective volume for minimal submanifolds with arbitrary codimension.

Suppose first that $\mathcal{M}$ is a hypersurface in $\mathbf{R}^n$. Then specifying a point $b \in \mathbf{R}^n \setminus x(M)$ we can introduce *the counting function* $\mathcal{N}(e,b)$ for the multiplicity of the radial projection relative to b, setting for any unit direction $e \in \mathbf{R}^n$

$$\mathcal{N}(e,b) = \sum_{a\in L_b(e)} a\#\mathcal{M} \equiv \#x^{-1}(L_b(e) \cap x(M)),$$

where $L_b(e)$ is a ray with the origin at b directed as e. The number $\mathcal{N}(e,b)$ can be interpreted as the multiplicity of the covering

$$\pi_b : \mathcal{M} \to S^{n-1}, \quad \pi_b(y) = \frac{y-b}{|y-b|}, \tag{15}$$

at a point e.

If $\operatorname{codim}\mathcal{M} > 1$, then the image of $\mathcal{M}$ after projection (15) is a null-measure subset in S^{n-1} and the second definition of $\mathcal{N}(e,b)$ is meaningless. Therefore we give the following generalization of the first definition.

Let $G_n^p(b)$ be the Grassman manifold of all nonoriented $(n-p)$-dimensional planes γ passing through b. Then $G_n^p(b)$ can be equipped with the unique Haar measure $d\gamma$ which is invariant under the action of the motion subgroup preserving b, and normalized by

$$\int_{G_n^p(b)} d\gamma = 1.$$

Let $R > 0$. By Sard's theorem we know that for $d\gamma$-almost all planes $\gamma \in G_n^p(b)$ the set of the preimages $x^{-1}(x(M) \cap \gamma \cap B_b(R))$ is a discrete one. Put

$$\mathcal{N}(b,\gamma;R) = \# x^{-1}(x(M) \cap \gamma \cap B_b(R)),$$

- the cardinality of the corresponding set. The quantity

$$\mathcal{N}(b;R) = \int_{G_n^p(b)} \mathcal{N}(b,\gamma;R)d\gamma$$

can now be interpreted as "the average multiplicity" of the intersection of $(n-p)$-dimensional planes with the part of $\mathcal{M}$ distant from b not further than R. Moreover, $\mathcal{N}(b;R)$ is an increasing function of R and hence there exists a finite or infinite limit

$$\mathcal{N}(b) = \lim_{R\to\infty} \mathcal{N}(b;R).$$

Lemma 2. *Let $\mathcal{M}$ be a p-dimensional properly immersed minimal surface in $\mathbf{R}^n$ without boundary and $b \notin \mathcal{M}$. Then*

$$Q(\mathcal{M}) \le \frac{1}{2}\mathcal{N}(b)\omega_{p+1}, \tag{16}$$

where ω_{p+1} is the p-dimensional Hausdorff measure of unit sphere S^p.

Proof. Without loss of generality we can assume that $b = 0$. We specify $R > 0$ and denote as above

$$M_0(R) = \{m \in M : |x(m)| < R\}.$$

We consider the composition

$$\sigma : \mathcal{M} \xrightarrow{x} \mathbf{R}^n \setminus \{0\} \xrightarrow{\pi} S^{n-1},$$

where π is defined as in (15) with $b = 0$. In order to find the Jacobian $\det(d\sigma)$ of the map σ at m we observe that

$$d\sigma_m = d\pi_{x(m)} \circ dx_m : T_m M \to T_{\sigma(m)} S^{n-1}.$$

By direct calculation one can show that

$$d\pi_a(X) = \frac{X - \pi(a)\langle X, \pi(a)\rangle}{|a|},$$

for all $a \in \mathbf{R}^n \setminus \{0\}$ and $X \in T_a\mathbf{R}^n$. Hence for any $Y \in T_m M$

$$d\sigma_m(Y) = \frac{Y - \overline{x}(m)\langle Y, \overline{x}(m)\rangle}{|x(m)|}$$

where $\overline{x}(m) = x(m)/|x(m)|$ and we identify Y with $dx_m(Y)$ and $T_m M$ with a subspace of $T_{x(m)}\mathbf{R}^n \cong \mathbf{R}^n$ through the isometry dx_m. Choose an orthonormal basis $Y_1, \dots, Y_p$ in $T_m M$. We then have

$$\det{}^2(d\sigma_m) = \langle w, w\rangle$$

where

$$w = d\sigma_m(Y_1) \wedge d\sigma_m(Y_2) \wedge \ldots \wedge d\sigma_m(Y_p) =$$

$$= |x|^{-p}\,(Y_1 - \overline{x}\langle Y_1, \overline{x}\rangle) \wedge \ldots \wedge (Y_p - \overline{x}\langle Y_p, \overline{x}\rangle)$$

$$= |x|^{-p}(Y_1 \wedge Y_2 \wedge \ldots \wedge Y_p - \sum_{i=1}^{p} Y_1 \wedge \ldots \wedge Y_{i-1} \wedge \overline{x} \wedge Y_{i+1} \wedge \ldots \wedge Y_p \langle \overline{x}, Y_i\rangle)$$

$$= \frac{Y_1 \wedge Y_2 \wedge \ldots \wedge Y_p}{|x|^p}\left(1 - \sum_{i=1}^{p} \langle Y_i, \overline{x}(m)\rangle^2\right)$$

$$-\sum_{i=1}^{p} Y_1 \wedge \ldots \wedge Y_{i-1} \wedge \overline{x}^{\perp} \wedge Y_{i+1} \wedge \ldots \wedge Y_p \langle \overline{x}, Y_i \rangle,$$

and consequently,

$$\det{}^2(d\sigma_m) = \frac{|\overline{x}^{\perp}(m)|^2}{|x(m)|^{2p}}.$$

Thus we obtain

$$|\det(d\sigma_m)| = \frac{|\overline{x}^{\perp}(m)|}{|x(m)|^{p}} = \frac{|x^{\perp}(m)|}{|x(m)|^{p+1}}, \tag{17}$$

the required expression for the Jacobian of $d\sigma_m$. By the change of coordinates formula we obtain from (17)

$$\int\limits_{M(R)} \frac{|x^{\perp}(m)|^2}{|x(m)|^{p+2}} \leq \int\limits_{M(R)} \frac{|x^{\perp}(m)|}{|x(m)|^{p+1}}$$

$$= \int\limits_{M(R)} |\det(d\sigma_m)| = \int\limits_{\sigma(M(R))} \chi(s) d\mathcal{H}^p(s), \tag{18}$$

where $\chi(s)$ is the cardinality of the preimage $\sigma^{-1}(s) \cap M(R)$ for the given $s \in \sigma(M(R)) \subset S^{n-1}$ and $\mathcal{H}^p$ is the corresponding Hausdorff measure on $\sigma(M(R))$. According to the theorem of Federer ([3], Theorem 3.2.48), we conclude that for every $\mathcal{H}^p$-measurable and $(\mathcal{H}^p, p)$-rectifiable set $F \subset S^{n-1}$ and positive summable function f on F

$$\int\limits_{s\in F} f(s) d\mathcal{H}^p(s) = \frac{\omega_{p+1}}{2} \int\limits_{\gamma \in G_n^p(0)} f^*(\gamma \cap F) d\gamma, \tag{19}$$

where $f^*(\gamma \cap F) = \sum_{s\in\gamma\cap F} f(s)$ is well-defined function for $d\gamma$-almost all planes $\gamma \in G_n^p(0)$. Then it follows from (18) and (19)

$$\int\limits_{M(R)} \frac{|x^{\perp}(m)|^2}{|x(m)|^{p+2}} \leq \frac{\omega_{p+1}}{2} \int\limits_{\gamma \in G_n^p(0)} \#\sigma^{-1}[\gamma \cap \sigma(M(R))] d\gamma = \frac{\omega_{p+1}}{2} \mathcal{N}(0, R),$$

and taking $R \to \infty$ we arrive at the required estimate (16).

Thus using the previous lemma and Corollary 2, we have

Corollary 3. *Let $\mathcal{M}$ be a properly immersed p-dimensional minimal surface in $\mathbb{R}^n$ without boundary. Suppose that for some point $b \in \mathbb{R}^n$ the cardinality of the set of intersection points (taking into account multiplicity) of any $\gamma \in G_n^p(b)$ and $x(M)$ does not exceed k. Then M is a manifold with finitely many ends and*

$$\ell(M) \leq kc_p,$$

where

$$c_p = 2^{p-1}(p+1)\sqrt{\pi}\Gamma(\frac{p+2}{2})\Gamma^{-1}(\frac{p+3}{2}) = \frac{2^{p-1}p\omega_{p+1}}{\omega_p}$$

and Γ is the Euler gamma-function and ω_{p+1} is as in Lemma 2.

Corollary 4. *Let $\mathcal{M}$ be a properly embedded p-dimensional minimal hypersurface without boundary. Assume that $\mathcal{M}$ is starlike with respect to some point in $\mathbb{R}^{p+1}$. Then the number of ends $\ell(M)$ satisfies*

$$\ell(M) \leq 2c_p,$$

where the constant c_p is from the previous lemma.

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