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On the Exponential Transform of Multi-Sheeted Algebraic Domains

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Abstract. We introduce multi-sheeted versions of algebraic domains and quadrature domains, allowing them to be branched covering surfaces over the Riemann sphere. The two classes of domains turn out to be the same, and the main result states that the extended exponential transform of such a domain agrees, apart from some simple factors, with the extended elimination function for a generating pair of functions. In an example we discuss the algebraic curves associated to level curves of the Neumann oval, and determine which of these give rise to multi-sheeted algebraic domains.

Keywords. Algebraic domain, quadrature domain, exponential transform, elimination function, Riemann surface, Klein surface, Neumann's oval.

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1. Introduction

Extending some previous work [26, 27, 28, 17, 18] on the rationality of the exponential transform here we go on to consider what we believe is the most general type of domains for which the exponential transform, in an extended sense, can be expected to have a "core" consisting of a rational function. Around this core there will then be "satellites" of some rather trivial factors, depending on the regimes at the locations of the independent variables.

The "domains" we consider will actually be covering surfaces over the Riemann sphere. The terminology "quadrature Riemann surface" has already been introduced by M. Sakai [32] for the type of domains in question. An alternative name could be "multi-sheeted algebraic domain", to extend a terminology used by A. Varchenko and P. Etingof [38]. In the present work we shall mostly use the

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latter terminology because we will not emphasize so much the quadrature properties, but rather take as a starting point the way the domains are generated, namely by a pair of meromorphic function on a compact Riemann surface.

Given a bounded domain $\Omega \subset \mathbb{C}$, the traditional exponential transform [4, 26, 27, 28, 16] of Ω is the function of two complex variables defined by

$$E_\Omega(z,w) = \exp\left(\frac{1}{2\pi \mathrm{i}}\int_\Omega \frac{d\zeta}{\zeta - z}\wedge\frac{d\bar\zeta}{\bar\zeta-\bar w}\right), \qquad z,w\in\mathbb{C}.$$

For the unit disk $\mathbb{D}$ it is (see [16]):

$$E(z,w) = \begin{cases} 1-\dfrac{1}{z\bar w}, & z,w\in\mathbb{C}\setminus\overline{\mathbb{D}},\\ 1-\dfrac{\bar z}{\bar w}, & z\in\mathbb{D},\ w\in\mathbb{C}\setminus\overline{\mathbb{D}},\\ 1-\dfrac{w}{z} & z\in\mathbb{C}\setminus\overline{\mathbb{D}},\ w\in\mathbb{D},\\ \dfrac{|z-w|^2}{1-z\bar w} & z,w\in\mathbb{D}.\end{cases} \tag{1}$$

Using the Schwarz function [8, 36] for $\partial\mathbb{D}$,

$$S(z)=\frac{1}{z},$$

(1) can be written

$$E(z,w) = \left(\frac{\bar z-\bar w}{S(z)-\bar w}\right)^{\rho(z)}\left(\frac{z-w}{z-\overline{S(w)}}\right)^{\rho(w)}\mathcal{E}(z,\bar w), \tag{2}$$

where $\mathcal{E}(z,\bar w)$ is the rational function

$$\mathcal{E}(z,\bar w) = \frac{z\bar w-1}{z\bar w}$$

and $\rho = \chi_{\mathbb{D}}$ is the characteristic function of $\mathbb{D}$. The expression (2) reveals the general structure of the exponential transform of any algebraic domain.

The function $\mathcal{E}(z,\bar w)$ is an instance of the elimination function, which can be defined by means of the meromorphic resultant $\mathcal{R}(f,g)$ of two meromorphic functions f and g. The resultant is defined as the multiplicative action of g on the divisor (f) of f, namely $\mathcal{R}(f,g) = g((f))$, and the elimination function is $\mathcal{E}_{f,g}(z,\bar w) = \mathcal{R}(f-z, g-\bar w)$. In the case of the unit disk, or any quadrature domain, the relevant elimination function which enters into the exponential transform is the one with $f(\zeta)=\zeta$, $g(\zeta)=S(\zeta)$. This means that in the present example we get

$$\begin{aligned}\mathcal{E}(z,\bar w) &= \mathcal{R}(\zeta - z, S(\zeta)-\bar w) = (S(\zeta)-\bar w)((\zeta - z))\\ &= (S(\zeta)-\bar w)(1\cdot(z) - 1\cdot(\infty)) = \frac{S(z)-\bar w}{S(\infty)-\bar w} = \frac{z\bar w-1}{z\bar w},\end{aligned}$$

as desired.

The aim of the present paper is to generalize the formula (2) as far as possible. This will involve an extended exponential transform in four complex variables and an analogous extended elimination function in four variables, defined in terms of a conjugate pair of meromorphic functions on a fairly general compact symmetric Riemann surface.

The paper is organized as follows: Sections 2 and 3 contain general preliminary material. In Section 4 we introduce the concepts of multi-sheeted algebraic domains and quadrature Riemann surfaces and prove that they are equivalent. The main result is stated in Section 5 and the proof is given in Section 6. Section 7, finally, is devoted to examples, namely the ellipse and Neumann's oval.

2. The Cauchy and exponential transforms

The Cauchy transform of a bounded density function ρ in $\mathbb{C}$ is

$$C_\rho(z) = \frac{1}{2\pi \mathrm{i}} \int \frac{\rho(\zeta) d\zeta \wedge d\bar{\zeta}}{\zeta - z}.$$

Typically the functions ρ which will appear in this paper will be like the characteristic function of a domain, or the corresponding integer valued counting function for a multi-sheeted domain. From C_ρ, the density ρ can be recovered by

$$\rho(z) = \frac{\partial C_\rho(z)}{\partial \bar{z}}, \tag{3}$$

to be interpreted in the sense of distributions.

If one writes the definition of the Cauchy transform as

$$C_\rho(z) = \frac{1}{2\pi \mathrm{i}} \int \rho(\zeta) \frac{d\zeta}{\zeta - z} \wedge d\bar{\zeta}$$

one realizes that it suffers from a certain lack of symmetry. A more balanced object would be the "double Cauchy transform",

$$C_\rho(z, w) = \frac{1}{2\pi \mathrm{i}} \int \rho(\zeta) \frac{d\zeta}{\zeta - z} \wedge \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}}. \tag{4}$$

In fact, this double transform is much richer than the original transform, and after exponentiation it gives the exponential transform, which is by now quite well studied (cf. [4, 26, 27, 28, 16]):

$$E_\rho(z, w) = \exp C_\rho(z, w).$$

The original Cauchy transform can be recovered as

$$C_\rho(z) = \mathrm{res}_{w=\infty} C_\rho(z, w) = -\lim_{w \to \infty} \bar{w} C_\rho(z, w),$$

at least if ρ vanishes in a neighborhood of infinity. One disadvantage with the double Cauchy transform is that the formula (3) turns into the more complicated

$$\frac{\partial C_\rho(z,w)}{\partial \bar{z}} = \frac{\rho(z)}{\bar{z}-\bar{w}}. \tag{5}$$

On the other hand we have the somewhat nicer looking

$$\frac{\partial^2 C_\rho(z,w)}{\partial \bar{z} \partial w} = -\pi \rho(z) \delta(z-w),$$

where δ denotes the Dirac distribution.

Now, even the double Cauchy transform is not entirely complete. It contains the Cauchy kernel $(\zeta - z)^{-1}\, d\zeta$, which is a meromorphic differential on the Riemann sphere with a pole at $\zeta = z$, but it also has a pole at $\zeta = \infty$. It is natural to make the latter pole visible and movable. That would have the additional advantage that one can avoid the two Cauchy kernels, which appear in the definitions of the double Cauchy transform and the exponential transform, having coinciding poles (namely at infinity). Thus we arrive naturally at the extended Cauchy and exponential transforms:

$$C_\rho(z,w;a,b) = \frac{1}{2\pi \mathrm{i}} \int \rho(\zeta) \left(\frac{d\zeta}{\zeta - z} - \frac{d\zeta}{\zeta - a} \right) \wedge \left(\frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}} - \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{b}} \right), \tag{6}$$

$$E_\rho(z,w;a,b) = \exp C_\rho(z,w;a,b) = \frac{E_\rho(z,w)E_\rho(a,b)}{E_\rho(z,b)E_\rho(a,w)}. \tag{7}$$

If the points z, w, a, b are taken to be all distinct, then both transforms are well defined and finite for any bounded density function ρ on the Riemann sphere. For example, with $\rho \equiv 1$, $E_\rho(z,w;a,b)$ turns out to be the modulus squared of the cross-ratio. See [17] for further details.

3. The resultant and the elimination function

Here we shall briefly review the definitions of the meromorphic resultant and the elimination function, as introduced in [17], referring to that paper for details. If f is a meromorphic function on any compact Riemann surface M we denote by (f) its divisor of zeros and poles, symbolically $(f) = f^{-1}(0) - f^{-1}(\infty)$. If D is any divisor and g is a meromorphic function we denote by $g(D)$ the multiplicative action of g on D. For example, if $D = 1 \cdot (a) + 1 \cdot (b) - 2 \cdot (c)$, $a, b, c \in M$, then $g(D) = g(a)g(b)/g(c)^2$. Now the meromorphic resultant between f and g is, by definition,

$$\mathcal{R}(f,g) = g((f))$$

whenever this makes sense.

The elimination function is

$$\mathcal{E}_{f,g}(z,w) = \mathcal{R}(f-z, g-w),$$

where $z, w \in \mathbb{C}$ are parameters. It is always a rational function in z and w, more precisely of the form

$$\mathcal{E}_{f,g}(z,w) = \frac{Q(z,w)}{P(z)R(w)}, \tag{8}$$

where Q, P and R are polynomials, and it embodies the polynomial relationship necessary (since M is compact) between f and g:

$$\mathcal{E}_{f,g}(f(\zeta), g(\zeta)) = 0, \qquad \zeta \in M.$$

We also have the extended elimination function, defined by

$$\mathcal{E}_{f,g}(z,w;a,b) = \mathcal{R}\left(\frac{f-z}{f-a}, \frac{g-w}{g-b}\right).$$

To relate the elimination function to the exponential transform one needs integral formulas for the elimination function. If f is meromorphic on M with divisor (f), let σ_f be a 1-chain such that $\partial\sigma_f = (f)$ and such that $\log f$ has a single-valued branch, which we denote $\operatorname{Log} f$, in $M \setminus \operatorname{supp}\sigma_f$. Then $\operatorname{Log} f$ can be viewed as a distribution on M, and its exterior differential in the sense of distributions (or currents) is

$$d \operatorname{Log} f = \frac{df}{f} - 2\pi \mathrm{i} dH_{\sigma_f}. \tag{9}$$

Here dH_{σ_f} is the 1-form current supported by σ_f and defined locally, away from $\partial\sigma_f$, as the differential (in the sense of currents) of that function H_{σ_f} which is $+1$ on the right-hand side of σ_f, zero on the left-hand side. Globally dH_{σ_f} is not exact (despite the notation), not even closed. To be precise,

$$d(dH_{\sigma_f}) = \frac{1}{2\pi\mathrm{i}} d\left(\frac{df}{f}\right) = \delta_{(f)} dx \wedge dy,$$

where $\delta_{(f)}$ denotes the finite distribution of point masses (or charges) corresponding to (f). We shall also need the fact that dH_{σ_f} has the period reproducing property

$$\int_M dH_{\sigma_f} \wedge \tau = \int_{\sigma_f} \tau, \tag{10}$$

holding for any smooth 1-form τ.

Now we have (essentially [17, Thm. 2])

$$\mathcal{E}_{f,g}(z,w;a,b) = \exp\left(\frac{1}{2\pi\mathrm{i}} \int_M \left(\frac{df}{f-z} - \frac{df}{f-a}\right) \wedge d \operatorname{Log} \frac{g-w}{g-b}\right). \tag{11}$$

The integrand is a 2-form current with support on the 1-chains σ_{g-w} and σ_{g-b} (because away from these curves the integrand contains $d\zeta \wedge d\zeta$), so the integral is rather a line integral than an area integral. In fact, the above can also be written as

$$\mathcal{E}_{f,g}(z,w;a,b) = \exp\left(\int_{\sigma_{g-w}} \left(\frac{df}{f-z} - \frac{df}{f-a}\right) - \int_{\sigma_{g-b}} \left(\frac{df}{f-z} - \frac{df}{f-a}\right)\right),$$

which perhaps clarifies the connection to the definition of the elimination function.

4. Multi-sheeted algebraic domains

The boundary of a quadrature domain (algebraic domain) is an algebraic curve, but by no means every algebraic curve arises in this way. However, the gap between the two classes of objects can be reduced considerably by extending the notion of a quadrature domain, allowing it to have several sheets and to be branched over the Riemann sphere. This will take essentially one half of all algebraic curves into the framework of quadrature domains and exponential transforms. One step in this direction was taken in Sakai [32], where a notion of quadrature Riemann surface was introduced in a special case. Below we shall take some further steps.

Let M be any symmetric compact (closed) Riemann surface. Slightly more generally, we shall allow M to be disconnected, namely to be a finite disjoint union of Riemann surfaces. The symmetry means that M is provided with an anticonformal involution $J\colon M \to M$, $J \circ J =$ identity. If M is disconnected then J is allowed to map one component of M onto another. Let Γ denote the set of fixed points of J. Simple examples of symmetric Riemann surfaces are $M = \mathbb{P}$ (the Riemann sphere) with the involution being either $J_1(\zeta) = 1/\bar{\zeta}$ or $J_2(\zeta) = -1/\bar{\zeta}$. In the first case $\Gamma = \{\zeta : |\zeta| = 1\}$ and $M \setminus \Gamma$ has two components, in the second case Γ is empty and hence $M \setminus \Gamma$ has only one component. One can also think of identifying the points ζ and $J(\zeta)$. The identification spaces become, in the first case ($J = J_1$) the unit disk together with its boundary, and in the second case ($J = J_2$) the projective plane, thus a non-orientable surface.

In general, the orbit space $N = M/J$, obtained by identifying ζ and $J(\zeta)$ for any $\zeta \in M$, is a Klein surface, possibly with boundary. A Klein surface [2] is defined in the same way as a Riemann surface except that it is allowed to be non-orientable and that both holomorphic and antiholomorphic transition functions between coordinates are allowed. The possible boundary points of N are those coming from Γ under the identification. From the Klein surface N, M can be recovered by a natural doubling procedure, described in [33, Sec. 2.2] and in [2], for example. The latter reference actually describes several types of doubles (the complex double, the orienting double and the Schottky double), but the description in [33] will be enough for our purposes. When N is orientable and has a boundary the doubling procedure gives what is usually called the Schottky double, named after the inventor of the idea, F. Schottky [34]. The idea was later extended to more general surfaces by F. Klein [9].

Continuing the discussion of M, J and N, if M is connected but $M \setminus \Gamma$ disconnected, then $M \setminus \Gamma$ has exactly two components, say M_+ and M_-, and J maps each of them onto the other. It then follows that $(M \setminus \Gamma)/J$ can be identified with M_+ (or M_-), and in particular that N is orientable, hence is (after choice of orientation) an ordinary Riemann surface with boundary. If $M \setminus \Gamma$ is connected then N necessarily is non-orientable.

It is relevant to allow M to have several components. For example, in Section 7 we will encounter the double of the Riemann sphere, which simply is two Riemann spheres with the opposite conformal structure and with J mapping one onto the other.

Now to the definition of "multi-sheeted algebraic domain". There are two ingredients. The first is a compact symmetric Riemann surface (M, J) such that M is connected and such that $M \setminus \Gamma$ has two components, one of which, call it M_+, is to be selected. We could equally well have started with M_+, to be any Riemann surface with boundary (bordered Riemann surface), and then let M be the double of M_+. The second ingredient is a non-constant meromorphic function f on M.

Definition 1. A *multi-sheeted algebraic domain* is a pair (M_+, f), where M_+ is a bordered Riemann surface and f is a non-constant meromorphic function on the double M of M_+. Two pairs, (M_+, f) and $(\tilde{M}_+, \tilde{f})$ are considered the same if there is a biholomorphic mapping $\phi\colon M \to \tilde{M}$ such that $\phi \circ J = \tilde{J} \circ \phi$ and $f = \tilde{f} \circ \phi$.

The equivalence simply means that it is the image $f(M_+)$, with appropriate multiplicities, which counts. Note that also (M_-, f) is a multi-sheeted algebraic domain, if (M_+, f) is. Trivial examples of a multi-sheeted algebraic domain are obtained by taking $M = \mathbb{P}$ and $J(\zeta) = 1/\bar{\zeta}$. Then with f any non-constant rational function $(\mathbb{D}, f)$ will be a multi-sheeted algebraic domain, as well as $(\mathbb{P} \setminus \overline{\mathbb{D}}, f)$. Some further examples will be discussed in Section 7.

Along with f, meromorphic on M, the symmetry J provides one more meromorphic function on M, namely

$$f^* = \overline{(f \circ J)}.$$

With $Q(z, w)$ the polynomial in (8) for f and $g = f^*$, the map

$$M \ni \zeta \mapsto (f(\zeta), f^*(\zeta))$$

parametrizes the curve $Q(z, \bar{w}) = 0$. This parametrization is one-to-one (except for finitely many points) if and only if f and f^* generate the field of meromorphic functions on M (form a primitive pair in the terminology of [10]). This will generically be the case if f is chosen "at random", but there are certainly many exceptions. For example, f may be already symmetric in itself, i.e., $f = f^*$, and then f and f^* is a primitive pair only if M is the Riemann sphere and f is a Möbius transformation. If f and f^* are not a primitive pair, then the polynomial $Q(z,w)$ in (8) is reducible.

In case M_+ is planar (i.e. is topologically equivalent to a planar domain) and f is univalent on M_+ without poles on $M_+ \cup \Gamma$, then $\Omega = f(M_+)$ is an ordinary algebraic domain, in other words a classical quadrature domain. In this case f and f^* do form a primitive pair (see [14]). The reference to "quadrature" can be explained in terms of a residue calculation. In the present generality it is natural to use the spherical metric on the Riemann sphere in place of the customary Euclidean metric. This will remove some integrability problems, and the point at infinity can be treated on the same footing as other points. Quadrature domains for the spherical measure have been previously discussed, at least in fluid dynamic contexts, for example in [38] (Hele-Shaw flow), [6] (vortex patches).

So let (M_+, f) be a multi-sheeted algebraic domain, and let h be a function holomorphic in a neighborhood of $M_+ \cup \Gamma$, to be used as a test function. The function f is allowed to have poles in $M_+ \cup \partial M$, but since the spherical metric is invariant under rotations of the Riemann sphere such poles will never cause problems. They are simply the points where f attains the value $\infty \in \mathbb{P}$, which is a no more special value than other values on $\mathbb{P}$. However, for computational purposes it is useful to work with $g = 1/f$ instead of f at places where f is large. Then we have, by exterior differentiation,

$$d\left(\frac{h\bar{f}df}{1+f\bar{f}}\right) = \frac{hd\bar{f}\wedge df}{(1+f\bar{f})^2} = \frac{hd\bar{g}\wedge dg}{(1+g\bar{g})^2} = d\left(\frac{h\bar{g}dg}{1+g\bar{g}}\right),$$

and where we can use either side depending on whether f is small or large. Dividing the Riemann sphere into the two pieces $|f| < R$ and $|f| > R$, where $0 < R < \infty$ is chosen so that there will be no residues on $|f| = R$ for the differential forms appearing below, we get the following computation:

$$\begin{aligned}
\frac{1}{2\pi \mathrm{i}}\int_{M_+} \frac{hd\bar{f}\wedge df}{(1+f\bar{f})^2} &= \frac{1}{2\pi \mathrm{i}}\int_{M_+\cap\{|f|<R\}} \frac{hd\bar{f}\wedge df}{(1+f\bar{f})^2} + \frac{1}{2\pi \mathrm{i}}\int_{M_+\cap\{|f|>R\}} \frac{hd\bar{f}\wedge df}{(1+f\bar{f})^2} \\
&= \frac{1}{2\pi \mathrm{i}}\int_{M_+\cap\{|f|<R\}} \frac{hd\bar{f}\wedge df}{(1+f\bar{f})^2} + \frac{1}{2\pi \mathrm{i}}\int_{M_+\cap\{|g|<1/R\}} \frac{hd\bar{g}\wedge dg}{(1+g\bar{g})^2} \\
&= \frac{1}{2\pi \mathrm{i}}\int_{\partial(M_+\cap\{|f|<R\})} \frac{h\bar{f}df}{1+f\bar{f}} + \frac{1}{2\pi \mathrm{i}}\int_{\partial(M_+\cap\{|g|<1/R\})} \frac{h\bar{g}dg}{1+g\bar{g}} \\
&= \sum_{M_+\cap\{|f|<R\}} \operatorname{res} \frac{hf^*df}{1+ff^*} + \sum_{M_+\cap\{|g|<1/R\}} \operatorname{res} \frac{hg^*dg}{1+gg^*}.
\end{aligned}$$

The residues above will, generally speaking, come from points where $1+ff^* = 0$, or equivalently $1 + gg^* = 0$. Poles of f (which simply appear as zeros of g) will not contribute, unless f^* vanishes at the point in question (in which case there usually will be a contribution). Similarly, poles of f^* will not contribute unless f

vanishes at the point. In any case, the right member above is of the form

$$L(h)=\sum_{k=1}^{m}\sum_{j=0}^{n_k-1}c_{kj}h^{(j)}(b_k),$$

i.e. equals the action on h by a distribution with support in finitely many points.

If f is univalent the above identity becomes an ordinary quadrature identity (although for the spherical measure) of the form

$$\frac{1}{2\pi \mathrm{i}}\int_\Omega g(z)\frac{d\bar{z}\wedge dz}{(1+|z|^2)^2}=\tilde{L}(g), \tag{12}$$

holding for all integrable analytic functions g in $\Omega = f(M_+)$. Here the right member is given by $\tilde{L}(g) = L(g \circ f)$, which still is the action of a distribution with finite support.

When f is not univalent one should still think of the quadrature identity in the same way as in (12), just with the difference that Ω is a region with several sheets over the Riemann sphere. The test functions should be allowed to take different values at points lying above one and the same point, but on different sheets. Thus those of the form $g \circ f$ (i.e. those which would become $g(z)$ in a formulation like (12)) are too special. This is most easily expressed by pulling everything back to M_+, in which case we simply have the identity originally obtained

$$\frac{1}{2\pi \mathrm{i}}\int_{M_+} h\frac{d\bar{f}\wedge df}{(1+|f|^2)^2}=L(h), \tag{13}$$

for h holomorphic in a neighborhood of $M_+\cup\Gamma$, and by approximation for functions h holomorphic and integrable (with respect to $(1/(2\pi\mathrm{i}))(1+|f|^2)^{-2}d\bar{f}\wedge df$) in M_+. An equivalent formulation is that there exists a positive divisor D in M_+ such that

$$\frac{1}{2\pi \mathrm{i}}\int_{M_+} h\frac{d\bar{f}\wedge df}{(1+|f|^2)^2}=0 \tag{14}$$

holds for every holomorphic and integrable function h in M_+ with $(h)\geq D$.

Definition 2. A *quadrature Riemann surface (for the spherical metric)* is a pair (M_+, f), where M_+ is a bordered Riemann surface and f is a non-constant meromorphic function on M_+ such that

$$\frac{1}{2\pi \mathrm{i}}\int_{M_+} \frac{d\bar{f}\wedge df}{(1+|f|^2)^2}<\infty$$

and such that (13) (or (14)) holds for some L (respectively D) and the classes of functions h indicated.

The notion of equivalence between pairs is the same as in Definition 1. If f is meromorphic on M then we have

$$\frac{1}{2\pi\mathrm{i}}\int_{M_+}\frac{d\bar{f}\wedge df}{(1+|f|^2)^2}\leq\frac{1}{2\pi\mathrm{i}}\int_{M}\frac{d\bar{f}\wedge df}{(1+|f|^2)^2}=\frac{n}{2\pi\mathrm{i}}\int_{\mathbb{P}}\frac{d\bar{z}\wedge dz}{(1+|z|^2)^2}=n<\infty,$$

where n is the order of f. Thus we have obtained one direction of the following result.

Proposition 1. *A pair (M_+, f) is a multi-sheeted algebraic domain if and only if it is a quadrature Riemann surface.*

Proof. It remains to prove that a quadrature Riemann surface is a multi-sheeted algebraic domain. In many special cases this has already been done (for the Euclidean metric), see for example [1, 31, 14, 32]. We shall discuss here the general case under the simplifying assumption that f is meromorphic in a neighborhood of $M_+ \cup \Gamma$.

It will be convenient to introduce some further notation. With $A \subset M$ any subset and D any divisor on M we denote by $\mathcal{O}_D(A)$ the set of functions h meromorphic in a neighborhood of A and with $(h) \geq D$. Similarly, $\mathcal{O}_D^{1,0}(A)$ denotes the set of 1-forms ω, meromorphic in a neighborhood of A and satisfying $(\omega) \geq D$. We shall also use standard notation for cohomology groups.

Now, returning to the previous residue calculation one realizes that what needs to show is that if

$$\frac{1}{2\pi\mathrm{i}}\int_{\partial M_+} h\frac{\bar{f}df}{1+f\bar{f}}=0 \tag{15}$$

holds for all $h \in \mathcal{O}_D(M_+\cup\Gamma)$, for some sufficiently large divisor D, then f extends to a meromorphic function on M. On Γ we have

$$\frac{\bar{f}df}{1+f\bar{f}}=\frac{f^*df}{1+ff^*},$$

where the right member is holomorphic in a neighborhood of Γ and can be viewed as representing an element in the cohomology group $H^1(M, \mathcal{O}_{-D}^{1,0})$. When D is strictly positive this group is trivial since, by Serre duality [35, 11],

$$H^1(M,\mathcal{O}_{-D}^{1,0})\cong H^0(M,\mathcal{O}_D)^*=\mathcal{O}_D(M)^*=0$$

(here X^* denotes the dual space of X), hence there exist $\omega_\pm \in \mathcal{O}_{-D}^{1,0}(M_\pm \cup \Gamma)$ such that

$$\frac{f^*df}{1+ff^*}=\omega_+-\omega_-$$

in a neighborhood of Γ.

Clearly

$$\frac{1}{2\pi\mathrm{i}}\int_{\partial M_+} h\omega_+=0$$

for $h \in \mathcal{O}_D(M_+ \cup \Gamma)$ since the integrand is holomorphic in M_+, hence (15) reduces to the statement that

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial M_+} h\omega_- = 0 \tag{16}$$

for $h \in \mathcal{O}_D(M_+ \cup \Gamma)$. At this point we may use a general duality theorem (also related to Serre duality) going back to the work of J. Silva [37] and G. Köthe [19], extended by A. Grothendieck [13] and, in the form we need given by C. Auderset [3]. It states that the bilinear form in h and ω_- defined by the left member of (16) induces (with the path of integration moved slightly into M_-) a non-degenerate pairing

$$\mathcal{O}_D(M_+ \cup \Gamma)/\mathcal{O}_D(M) \times \mathcal{O}^{1,0}_{-D}(M_-)/\mathcal{O}^{1,0}_{-D}(M) \to \mathbb{C},$$

which exhibits each of the quotient spaces as the dual space of the other.

In view of this (16) implies that $\omega_- \in \mathcal{O}^{1,0}_{-D}(M)$, in particular that

$$\frac{f^* df}{1 + ff^*} = \omega_+ - \omega_- \in \mathcal{O}^{1,0}_{-D}(M_+ \cup \Gamma). \tag{17}$$

Since f is meromorphic in a neighborhood of $M_+ \cup \Gamma$, (17) implies that also f^* is meromorphic there, hence that f actually is meromorphic on all M, as was to be shown.

The usage of the general duality theorems above can be replaced by more direct arguments, like applying (13) to suitable Cauchy kernels. Specifically we may choose, with $\zeta \in M_-$,

$$h(z) = \Phi(z, \zeta; z_0, \zeta_0)\, d\zeta,$$

where the right member is a kernel which in the case of the Riemann sphere is the usual Cauchy kernel

$$\Phi(z, \zeta; z_0, \zeta_0)\, d\zeta = \frac{d\zeta}{\zeta - z} - \frac{d\zeta}{\zeta - z_0},$$

and which has counterparts with good enough properties on all compact Riemann surfaces (see [29]). The point ζ_0 is needed in higher genus. Actually the duality theorem discussed above can be proved using this kernel.

■

5. Statement of the main result

Let M, M_+, $\Gamma = \partial M_+$, f be as in Section 4, more precisely such that (M_+, f) is a multi-sheeted algebraic domain. To account for the multiplicities of $f(M_+)$ as a covering of the Riemann sphere we introduce the integer-valued counting function, or mapping degree,

$$\rho(z) = \mathrm{card}\{\zeta \in M_+ \colon f(\zeta) = z\},$$

pointwise well-defined for $z \in \mathbb{P} \setminus f(\Gamma)$. It is understood that points ζ are counted with the appropriate multiplicities. Set also

$$S = \overline{f \circ J \circ f^{-1}} = f^* \circ f^{-1}. \tag{18}$$

This is a multi-valued algebraic function in the complex plane which contains all local Schwarz functions of $f(\Gamma)$, because for $z \in f(\Gamma)$ one of the values of $S(z)$ is $\bar{z}$.

We will have to make expressions like $(S(z) - \bar{w}))^{\rho(z)}$ well-defined, i.e. single-valued, despite $S(z)$ itself being multi-valued. The natural definition is the following:

$$(S(z) - \bar{w})^{\rho(z)} = (f^* - \bar{w})((f - z)|_{M_+}).$$

Here $(f-z)|_{M_+}$ denotes the restriction of the divisor $(f-z)$ to M_+ and the right member then is the multiplicative action of $f^* - \bar{w}$ on $(f-z)|_{M_+}$. To spell it out, let

$$f^{-1}(z) \cap M_+ = \{\zeta_1, \dots, \zeta_{\rho(z)}\},$$

with repetitions according to multiplicities. Then

$$(S(z) - \bar{w})^{\rho(z)} = (f^*(\zeta_1) - \bar{w}) \cdot \dots \cdot (f^*(\zeta_{\rho(z)}) - \bar{w}), \tag{19}$$

which is a natural definition in view of (18). Clearly $(S(z) - \bar{w})^{\rho(z)}$ is an analytic function of z in regions where $\rho(z)$ is constant.

Now, for the main result, we have two functions which we want to relate to each other: one is the weighted exponential transform

$$\begin{aligned} E_\rho(z, w; a, b) &= \exp\Big(\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{P}} \rho(\zeta) \left(\frac{d\zeta}{\zeta - z} - \frac{d\zeta}{\zeta - a}\right) \wedge \left(\frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}} - \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{b}}\right)\Big) \\ &= \exp\Big(\frac{1}{2\pi \mathrm{i}} \int_{M_+} \left(\frac{df}{f - z} - \frac{df}{f - a}\right) \wedge \left(\frac{d\bar{f}}{\bar{f} - \bar{w}} - \frac{d\bar{f}}{\bar{f} - \bar{b}}\right)\Big), \end{aligned}$$

which can be viewed as a kind of potential of ρ, and the other is the elimination function, which is defined by algebraic means and always is a rational function, namely of the form

$$\mathcal{E}_{f,f^*}(z, \bar{w}; a, \bar{b}) = \mathcal{R}(\frac{f - z}{f - a}, \frac{f^* - \bar{w}}{f^* - \bar{b}}) = \frac{Q(z, \bar{w})Q(a, \bar{b})}{Q(z, \bar{b})Q(a, \bar{w})}. \tag{20}$$

The latter expression comes from (8) together with the observation that the one variable polynomials cancel in the four variable case. Note the cross-ratio-like structure, similar to that in (7). The nature of $E_\rho(z, w, a, b)$ depends on the locations of the points z, w, a, b, more precisely on the the values of ρ at these points. The main result is the following:

Theorem 2. *Let (M_+, f) be a multi-sheeted algebraic domain. Then, in the above notation,*

$$\begin{aligned}E_\rho(z,w;a,b) = \mathcal{E}_{f,f^*}(z,\bar{w};a,\bar{b})\\ \cdot\left(\frac{\bar{z}-\bar{w}}{S(z)-\bar{w}}\right)^{\rho(z)}\left(\frac{w-z}{\overline{S(w)}-z}\right)^{\rho(w)}\\ \cdot\left(\frac{\bar{a}-\bar{b}}{S(a)-\bar{b}}\right)^{\rho(a)}\left(\frac{b-a}{\overline{S(b)}-a}\right)^{\rho(b)}\\ \cdot\left(\frac{S(z)-\bar{b}}{\bar{z}-\bar{b}}\right)^{\rho(z)}\left(\frac{\overline{S(w)}-a}{w-a}\right)^{\rho(w)}\\ \cdot\left(\frac{S(a)-\bar{w}}{\bar{a}-\bar{w}}\right)^{\rho(a)}\left(\frac{\overline{S(b)}-z}{b-z}\right)^{\rho(b)}.\end{aligned}$$

6. Proof of the main result

Extending (5) to four variables gives

$$\frac{\partial C_\rho(z,w;a,b)}{\partial \bar{z}} = \frac{\rho(z)}{\bar{z}-\bar{w}} - \frac{\rho(z)}{\bar{z}-\bar{b}},$$

which says that the function

$$C_\rho(z,w;a,b) + \rho(z)\log\frac{\bar{z}-\bar{b}}{\bar{z}-\bar{w}}$$

is analytic in z in regions where $\rho(z)$ is constant, namely in each component of $\mathbb{P}\setminus f(\Gamma)$. Hence so is the exponential of it, namely

$$E_\rho(z,w;a,b)\cdot\left(\frac{\bar{z}-\bar{b}}{\bar{z}-\bar{w}}\right)^{\rho(z)}.$$

Augmenting this we have that the function

$$F(z) = E_\rho(z,w;a,b)\cdot\left(\frac{\bar{z}-\bar{b}}{\bar{z}-\bar{w}}\cdot\frac{S(z)-\bar{w}}{S(z)-\bar{b}}\right)^{\rho(z)}$$

is also analytic in $\mathbb{P}\setminus f(\Gamma)$, away from poles caused by the presence of $S(z)$. Now we claim that it is even better than that: $F(z)$ is meromorphic everywhere, hence is a rational function. To prove this it is enough to prove that $F(z)$ is continuous across $f(\Gamma)$. It is well-known that $E_\rho(z,w;a,b)$ is continuous in z (see for example [16]), so we only have to bother about the other factor. But it is easy to show that also this is continuous: spelling out as in (19) and assuming for example that $\rho(z)$ increases by one unit as $f(\Gamma)$ is crossed at a certain place we find that the factor

$$\left(\frac{S(z)-\bar{w}}{\bar{z}-\bar{w}}\right)^{\rho(z)}$$

changes from

$$\frac{(f^*(\zeta_1)-\bar{w})\cdots(f^*(\zeta_{\rho(z)})-\bar{w})}{(\overline{f(\zeta_1)}-\bar{w})\cdots(\overline{f(\zeta_{\rho(z)})}-\bar{w})}$$

to

$$\frac{(f^*(\zeta_1)-\bar{w})\cdots(f^*(\zeta_{\rho(z)})-\bar{w})(f^*(\zeta_{\rho(z)+1})-\bar{w})}{(\overline{f(\zeta_1)}-\bar{w})\cdots(\overline{f(\zeta_{\rho(z)})}-\bar{w})(\overline{f(\zeta_{\rho(z)+1})}-\bar{w})},$$

which clearly is a continuous change since the new point $\zeta_{\rho(z)+1}$ starts up on Γ. We proceed similarly for the factor

$$\left(\frac{\bar{z}-\bar{b}}{S(z)-\bar{b}}\right)^{\rho(z)}.$$

Repeating the above argument for w, a, b it follows that the function

$$\begin{aligned} E_\rho(z,w;a,b)\cdot\left(\frac{\bar{z}-\bar{b}}{\bar{z}-\bar{w}}\cdot\frac{S(z)-\bar{w}}{S(z)-\bar{b}}\right)^{\rho(z)}&\cdot\left(\frac{w-a}{w-z}\cdot\frac{\overline{S(w)}-z}{\overline{S(w)}-a}\right)^{\rho(w)}\\ \cdot\left(\frac{\bar{a}-\bar{w}}{\bar{a}-\bar{b}}\cdot\frac{S(a)-\bar{b}}{S(a)-\bar{w}}\right)^{\rho(a)}&\cdot\left(\frac{b-z}{b-a}\cdot\frac{\overline{S(b)}-a}{\overline{S(b)}-z}\right)^{\rho(b)}\end{aligned}$$

is rational in the variables z $\bar{w}$, a, $\bar{b}$. Thus, since also $\mathcal{E}_{f,f^*}(z,\bar{w};a,\bar{b})$ is rational in these variables, it is enough to prove that the formula in the statement of the theorem holds just locally, somewhere. We may then choose z, w, a, b close to each other, so that in particular $\rho(z)=\rho(w)=\rho(a)=\rho(b)$. In addition we may assume that this value is the smallest value of ρ occurring on $\mathbb{P}$. The case that it is zero can be treated exactly as in the proof of [17, Thm. 6], which concerns the special case $a=b=\infty$.

So let us for example assume that $\rho(z)=\rho(w)=\rho(a)=\rho(b)=1$ (the general case will be similar). Thus f attains the values z, w, a, b exactly once in M_+, and these four points on the Riemann sphere are close to each other. Let γ be an arc in $\mathbb{P}$ from b to w (e.g. the geodesic arc). Then the function

$$z\mapsto\log\frac{z-w}{z-b}$$

has a single-valued branch, call it $\operatorname{Log}\frac{z-w}{z-b}$, in $\mathbb{P}\setminus\gamma$, hence $\operatorname{Log}\frac{f-w}{f-z}$ is single-valued in $M\setminus f^{-1}(\gamma)$. We may consider $f^{-1}(\gamma)$ as a 1-chain, and as such it has the same role for $\operatorname{Log}\frac{z-w}{z-b}$ as σ_f has for $\operatorname{Log} f$ in (9), so that

$$d\operatorname{Log}\frac{f-w}{f-b}=\frac{df}{f-w}-\frac{df}{f-b}-2\pi\mathrm{i}\,dH_{f^{-1}(\gamma)}.$$

If f has degree n, then $f^{-1}(\gamma)$ consists of n small arcs, one of which is located on M_+. Let $\sigma = f^{-1}(\gamma) \cap M_+$ be that arc and let $\tilde{\sigma} = J(\sigma)$ be the arc reflected in M_-. In what follows we shall use f^{-1} in the restricted sense $f^{-1} = (f|_{M_+})^{-1}$. Thus

$$\partial\sigma = f^{-1}(w) - f^{-1}(b) = \left(\frac{f-w}{f-b}\right)\bigg|_{M_+},$$
$$\partial\tilde{\sigma} = \widetilde{f^{-1}(w)} - \widetilde{f^{-1}(b)} = \left(\frac{f^*-\bar{w}}{f^*-\bar{b}}\right)\bigg|_{M_-}.$$

Using (11), (9) and (10) we now get

$$\begin{aligned}\mathcal{E}_{f,f^*}(z,\bar{w};a,\bar{b}) &= \exp\left(\frac{1}{2\pi\mathrm{i}}\int_M \left(\frac{df}{f-z} - \frac{df}{f-a}\right) \wedge d\,\mathrm{Log}\,\frac{f^*-\bar{w}}{f^*-\bar{b}}\right) \\ &= \exp\left(\frac{1}{2\pi\mathrm{i}}\int_{M_+} \left(\frac{df}{f-z} - \frac{df}{f-a}\right) \wedge d\,\mathrm{Log}\,\frac{f^*-\bar{w}}{f^*-\bar{b}}\right) \\ &\quad \cdot \exp\left(\int_{\tilde{\sigma}} \left(\frac{df}{f-z} - \frac{df}{f-a}\right)\right).\end{aligned}$$

Here we start by rewriting the last factor according to

$$\begin{aligned}\exp\left(\int_{\tilde{\sigma}} \left(\frac{df}{f-z} - \frac{df}{f-a}\right)\right) &= \left[\frac{f-z}{f-a}\right]_{\widetilde{f^{-1}(b)}}^{\widetilde{f^{-1}(w)}} \\ &= \frac{\overline{f^*(f^{-1}(w))} - z}{\overline{f^*(f^{-1}(w))} - a} \cdot \frac{\overline{f^*(f^{-1}(b))} - a}{\overline{f^*(f^{-1}(b))} - z} \\ &= \frac{\overline{S(w)} - z}{\overline{S(w)} - a} \cdot \frac{\overline{S(b)} - a}{\overline{S(b)} - z}.\end{aligned}$$

Note that functions like $\overline{S(w) - z} = \overline{(S(w) - \bar{z})^{\rho(w)}}$ are single-valued in the present case (cf. (19)).

Next, the first factor can be integrated partially, to become

$$\begin{aligned}&\exp\left(\frac{1}{2\pi\mathrm{i}}\int_{M_+} \left(\frac{df}{f-z} - \frac{df}{f-a}\right) \wedge d\ \mathrm{Log}\,\frac{f^*-\bar{w}}{f^*-\bar{b}}\right) \\ &= \exp\left(-\frac{1}{2\pi\mathrm{i}}\int_{\partial M_+} \left(\frac{df}{f-z} - \frac{df}{f-a}\right) \mathrm{Log}\,\frac{f^*-\bar{w}}{f^*-\bar{b}}\right) \\ &\quad \cdot \exp\left(\frac{1}{2\pi\mathrm{i}}\int_{M_+} d\left(\frac{df}{f-z} - \frac{df}{f-a}\right) \mathrm{Log}\,\frac{f^*-\bar{w}}{f^*-\bar{b}}\right) \\ &= \exp\left(-\frac{1}{2\pi\mathrm{i}}\int_{\partial M_+} \left(\frac{df}{f-z} - \frac{df}{f-a}\right) \mathrm{Log}\,\frac{f^*-\bar{w}}{f^*-\bar{b}}\right) \\ &\quad \cdot \exp\left(\int_{M_+} \left(\delta_{f^{-1}(z)} - \delta_{f^{-1}(a)}\right)\, dxdy\ \mathrm{Log}\,\frac{f^*-\bar{w}}{f^*-\bar{b}}\right)\end{aligned}$$

$$= \exp\left(-\frac{1}{2\pi\mathrm{i}}\int_{\partial M_+}\left(\frac{df}{f-z}-\frac{df}{f-a}\right)\operatorname{Log}\frac{f^*-\bar{w}}{f^*-\bar{b}}\right)$$
$$\cdot\frac{S(z)-\bar{w}}{S(z)-\bar{b}}\cdot\frac{S(a)-\bar{b}}{S(a)-\bar{w}}.$$

We have to rework all expressions which are not yet in the form appearing in the statement of the theorem. So we next turn our attention to the first factor in the last expression obtained. This can be rewritten as

$$\exp\left(-\frac{1}{2\pi\mathrm{i}}\int_{\partial M_+}\left(\frac{df}{f-z}-\frac{df}{f-a}\right)\operatorname{Log}\frac{f^*-\bar{w}}{f^*-\bar{b}}\right)$$
$$= \exp\left(-\frac{1}{2\pi\mathrm{i}}\int_{\partial M_+}\left(\frac{df}{f-z}-\frac{df}{f-a}\right)\operatorname{Log}\frac{\bar{f}-\bar{w}}{\bar{f}-\bar{b}}\right)$$
$$= \exp\left(-\frac{1}{2\pi\mathrm{i}}\int_{M_+} d\left(\frac{df}{f-z}-\frac{df}{f-a}\right)\operatorname{Log}\frac{\bar{f}-\bar{w}}{\bar{f}-\bar{b}}\right)$$
$$\cdot\exp\left(\frac{1}{2\pi\mathrm{i}}\int_{M_+}\left(\frac{df}{f-z}-\frac{df}{f-a}\right)\wedge d\operatorname{Log}\frac{\bar{f}-\bar{w}}{\bar{f}-\bar{b}}\right)$$
$$= \exp\left(-\int_{M_+}\left(\delta_{f^{-1}(z)}-\delta_{f^{-1}(a)}\right)dxdy\operatorname{Log}\frac{\bar{f}-\bar{w}}{\bar{f}-\bar{b}}\right)$$
$$\cdot\exp\left(\frac{1}{2\pi\mathrm{i}}\int_{M_+}\left(\frac{df}{f-z}-\frac{df}{f-a}\right)\wedge\left(\frac{d\bar{f}}{\bar{f}-\bar{w}}-\frac{d\bar{f}}{\bar{f}-\bar{b}}\right)\right)$$
$$\cdot\exp\left(\int_{M_+}\left(\frac{df}{f-z}-\frac{df}{f-a}\right)d\overline{H_\sigma}\right)$$
$$= \frac{\bar{z}-\bar{b}}{\bar{z}-\bar{w}}\cdot\frac{\bar{a}-\bar{w}}{\bar{a}-\bar{b}}$$
$$\cdot\exp\left(\frac{1}{2\pi\mathrm{i}}\int_{M_+}\left(\frac{df}{f-z}-\frac{df}{f-a}\right)\wedge\left(\frac{d\bar{f}}{\bar{f}-\bar{w}}-\frac{d\bar{f}}{\bar{f}-\bar{b}}\right)\right)$$
$$\cdot\exp\left(-\int_{\sigma}\left(\frac{df}{f-z}-\frac{df}{f-a}\right)\right)$$
$$= \frac{\bar{z}-\bar{b}}{\bar{z}-\bar{w}}\cdot\frac{\bar{a}-\bar{w}}{\bar{a}-\bar{b}}\cdot E_\rho(z,w;a,b)\cdot\frac{w-a}{w-z}\cdot\frac{b-a}{b-z}.$$

Now putting all the pieces together we obtain the formula in the statement of the theorem.

7. Examples

7.1. The ellipse. Let D denote the domain inside the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

with semiaxes $a > b > 0$ and foci $\pm c = \pm\sqrt{a^2 - b^2}$. The exterior domain $\mathbb{P} \setminus \overline{D}$ is known to be a null quadrature domain [30] for the Euclidean metric. For the spherical metric it is a two point quadrature domain. Indeed, the Schwarz function for the ellipse is

$$S(z) = \frac{a^2 + b^2}{c^2} z \pm \frac{2ab}{c^2}\sqrt{z^2 - c^2},$$

and for h holomorphic in $\mathbb{P} \setminus \overline{D}$ and smooth up to the boundary we have

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{P}\setminus\overline{D}} h(z) \frac{d\bar{z} \wedge dz}{(1 + |z|^2)^2} = \sum \operatorname{res}_{z\in\mathbb{C}\setminus\overline{D}} h(z) \frac{S(z)dz}{1 + zS(z)}.$$

In replacing $\mathbb{P} \setminus \overline{D}$ (in the left member) by $\mathbb{C} \setminus \overline{D}$ (in the right member) we have taken into account that $S(z) \neq 0$ at the pole of z (namely at $z = \infty$). Indeed, we have $S(\infty) = \infty$ for both branches. According to the reasoning in the derivation of (12), this means that the pole of $S(z)/(1 + zS(z))\, dz$ at $z = \infty$ does not contribute to the integral in the left member. (This can also be seen directly, by first making the change of variable $z = 1/w$ in the left member.) Thus the residues come from the zeros of $1 + zS(z)$ in the exterior of the ellipse, and straight-forward computations show that there are exactly two such zeros, located on the imaginary axis and more precisely given by

$$z = \pm z_0 = \pm\frac{1}{\mathrm{i}c}\sqrt{a^2 + b^2 + 2a^2b^2 + 2ab\sqrt{1 + a^2 + b^2 + a^2b^2}}.$$

Thus we have a quadrature identity of the form

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{P}\setminus\overline{D}} h(z) \frac{d\bar{z} \wedge dz}{(1 + |z|^2)^2} = c_0[h(z_0) + h(-z_0)], \tag{21}$$

c_0 being the residue of $S(z)/(1 + zS(z))\, dz$ at $z = \pm z_0$.

The exterior of the ellipse is the conformal image of the unit disk under the Joukowski map

$$f(\zeta) = \frac{c^2\zeta^2 + (a + b)^2}{2(a + b)\zeta}.$$

Thus $(\mathbb{D}, f)$, or simply $(\mathbb{P} \setminus \overline{D}, z)$ with z denoting the identity function, is a (single-sheeted) algebraic domain. The same function f maps the exterior of the unit disk onto a multi-sheeted algebraic domain, i.e. $(\mathbb{P} \setminus \overline{\mathbb{D}}, f)$ is (or represents) such a domain. It covers D twice and $\mathbb{P} \setminus \overline{D}$ once, in other words the counting function is

$$\rho(z) = \begin{cases} 2 & \text{for } z \in D, \\ 1 & \text{for } z \in \mathbb{P} \setminus \overline{D}. \end{cases}$$

Since there are several sheets the associated quadrature identity is best expressed in a form pulled-back to $\mathbb{P} \setminus \overline{\mathbb{D}}$, i.e. on the form (13). A slightly weaker form is

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{P}} \rho(z) h(z) \frac{d\bar{z} \wedge dz}{(1 + |z|^2)^2} = c_1[h(z_1) + h(-z_1)],$$

where

$$\pm z_1 = \pm\frac{1}{\mathrm{i}c}\sqrt{a^2+b^2+2a^2b^2-2ab\sqrt{1+a^2+b^2+a^2b^2}}$$

and c_1 is the residue of $S(z)/(1+zS(z))\,dz$ at $z=\pm z_1$. This form is weaker because it only uses test functions $h(z)$ that take the same values on the two sheets over D.

7.2. Neumann's oval: classification of level curves. By inversion in the unit circle and a rotation by 90 degrees (for convenience) the ellipse transforms into a curve known as Neumann's oval [24, 25, 36, 20] with equation

$$a^2b^2(x^2+y^2)^2 - a^2x^2 - b^2y^2 = 0. \tag{22}$$

The exterior of the ellipse transforms into a bounded domain Ω, and since inversions and rotations are rigid transformations with respect to the spherical measure, also Ω will be a two point quadrature domain for the spherical measure. The formula is immediately obtained by inversion and rotation of (21). The domain Ω also satisfies a quadrature identity for the Euclidean measure (indeed, both types of quadrature identities are equivalent to $S(z)$ being meromorphic in Ω). The latter quadrature identity is somewhat simpler, namely

$$\frac{1}{2\pi\mathrm{i}}\int_\Omega h\,d\bar{z}dz = \frac{a^2+b^2}{4a^2b^2}\left[h\Big(-\frac{c}{2ab}\Big)+h\Big(\frac{c}{2ab}\Big)\right].$$

It turns out to be a quite rewarding task to investigate the algebraic curves corresponding to the level curves of the left member in (22), and in particular to determine which of them correspond to multi-sheeted algebraic domains. This task is what we are going to undertake for the remainder of this section. Most types of phenomena which could possibly show up really do show up among these curves.

In order to simplify a little we first scale so that the quadrature nodes above become ± 1. This means that $2ab=c$. Then we need only one parameter (in place of the two, a and b), which we take to be

$$r = \frac{\sqrt{a^2+b^2}}{c} > 1.$$

The quadrature identity now becomes

$$\int_\Omega h\,dxdy = \pi r^2[h(-1)+h(1)], \tag{23}$$

holding for all integrable analytic functions h in Ω. Set

$$Q(z,w) = z^2w^2 - z^2 - w^2 - 2r^2zw,$$

which is the polarized version ($(z,\bar{z})$ polarizes into (z,w)) of the left member in (22). There are exactly two open sets for which the quadrature identity (23) holds, namely

$$\Omega = \{z\in\mathbb{C}\colon Q(z,\bar{z})<0\}$$

and

$$[\Omega] = \{z \in \mathbb{C} \colon Q(z, \bar{z}) < 0\} \cup \{0\}.$$

The latter is just the completion of the former with respect to one missing point. The domain Ω (or $[\Omega]$) can be viewed as two disks glued together, or "smashed", or "added", and has been studied by many authors, see for example [5], [7, 20, 21].

To analyse the level curves of $Q(z, \bar{z})$ we set, for any $\alpha \in \mathbb{R}$,

$$Q_\alpha(z, w) = z^2w^2 - z^2 - w^2 - 2r^2zw - \alpha. \tag{24}$$

Let

$$Q_\alpha(t, z, w) = z^2w^2 - z^2t^2 - w^2t^2 - 2r^2zwt^2 - \alpha t^4$$

be the corresponding homogenous polynomial. We shall keep $r > 1$ fixed and just vary α. The real locus of Q_α in $\mathbb{C}$ is

$$\operatorname{loc}_{\mathbb{R}} Q_\alpha = \{z \in \mathbb{C} \colon Q(z, \bar{z}) = \alpha\}.$$

It represents the intersection with $\{w = \bar{z}\}$ ("the real") of the complex locus in $\mathbb{C}^2$,

$$\operatorname{loc}_{\mathbb{C}}\, Q_\alpha = \{(z, w) \in \mathbb{C}^2 \colon Q(z, w) = \alpha\},$$

which has a natural completion in the projective space $\mathbb{P}_2(\mathbb{C})$ as

$$\operatorname{loc} Q_\alpha = \operatorname{loc}_{\mathbb{P}_2(\mathbb{C})}\, Q_\alpha = \{(t : z : w) \in \mathbb{P}_2(\mathbb{C}) \colon Q_\alpha(t, z, w) = 0\}.$$

Here $\mathbb{C}^2$ is embedded in $\mathbb{P}_2(\mathbb{C})$ so that (z, w) corresponds to $(1 : z : w)$. By real points in $\mathbb{C}^2$ we mean points (z, w) satisfying $w = \bar{z}$, and these are the fixed points of the involution $J \colon (z, w) \mapsto (\bar{w}, \bar{z})$. In projective coordinates the involution is $J : (t : z : w) \mapsto (\bar{t} : \bar{w} : \bar{z})$. In addition to the anticonformal involution J, the curve $\operatorname{loc}_{\mathbb{C}} Q_\alpha$ has the conformal symmetries $(z, w) \mapsto (w, z)$ and $(z, w) \mapsto (-z, -w)$.

Let (M_α, J_α) be the compact symmetric Riemann surface corresponding to the locus $\operatorname{loc} Q_\alpha, J)$. As point sets they are identical except for a few singular points on $\operatorname{loc} Q_\alpha$, which are resolved on M_α. The real locus of Q_α corresponds to the fixed point set Γ_α of J_α.

A first observation is that the function $z \mapsto Q(z, \bar{z})$ has five stationary points in the complex plane. There are two global minima, on the level $\alpha = -(r^2 + 1)^2$, there are two saddle points on the level $\alpha = -(r^2 - 1)^2$ and there is one local maximum (at the origin), on the level $\alpha = 0$. These three values of α will correspond to changes of regime for the algebraic curve $Q_\alpha(z, w) = 0$.

Solving the equation $Q_\alpha(z, w) = 0$ for w as a function of z gives the Schwarz functions for the curves in the real locus. The result is

$$w = S_\alpha(z) = \frac{1}{z^2 - 1}\left(r^2z \pm \sqrt{z^4 + (r^4 - 1 + \alpha)z^2 - \alpha}\right). \tag{25}$$

We see that $S_\alpha(z)$ in general has four branch points. The levels α at which $Q(z, \bar{z})$ has stationary points are exactly those values of α for which some or all of these branch points resolve: for $\alpha = -(r^2 \pm 1)^2$ the square root resolve completely into second order polynomials, and for $\alpha = 0$ one pair of branch points resolves.

Let $p = p(\alpha)$ denote the genus of M_α, or equivalently of $\operatorname{loc} Q_\alpha$. The degree of Q_α is four, hence the genus formula in algebraic geometry [12, 23] tells that

$$p + s = \frac{(4-1)\cdot(4-2)}{2} = 3,$$

where $s \geq 0$ is a certain number related to the singular points. An analysis, carried out in detail in [15], shows that $\operatorname{loc} Q_\alpha$ passes through the points $(0:1:0)$ and $(0:0:1)$ and that at each of these points it has two simple cusps of multiplicity one with distinct tangent directions. In particular, the points $(0:1:0)$ and $(0:0:1)$ are singular, and it turns out that each of them gives the contribution $+1$ to s. Except for the above two points of infinity, $\operatorname{loc} Q_\alpha$ stays in $\mathbb{C}^2$. Thus, what remains of the genus formula is

$$p + e = 1, \tag{26}$$

where e denotes the contribution to s which comes from finite singular points. By (26), $e \leq 1$, so there is at most one finite singular point, and this must be visible in the real because non-real singular points necessarily come in pairs.

The above analysis preassumes that the curve $\operatorname{loc} Q_\alpha$, or polynomial Q_α, is irreducible. This is the case for most values of α, but there are two exceptions:

(i) For $\alpha = -(r^2-1)^2$, Q_α factors as

$$Q_\alpha(z,w) = [(z+1)(w+1) - r^2][(z-1)(w-1) - r^2].$$

Each factor defines its own algebraic curve and Riemann surface. These have genus zero and are moreover symmetric: the zero locus of each factor is preserved under the involution $(z,w) \mapsto (\bar{w}, \bar{z})$. The real locus $\operatorname{loc}_{\mathbb{R}} Q_\alpha$ is the union of two intersecting circles, those of radius r and centers ± 1. The intersection points, $z = \pm\sqrt{r^2-1}$ are saddle points for the function $z \mapsto Q(z,\bar{z})$.

(ii) For $\alpha = -(r^2+1)^2$, Q_α factors as

$$Q_\alpha(z,w) = [(z+1)(w-1) - r^2][(z-1)(w+1) - r^2],$$

where again each factor defines its own algebraic curve and Riemann surface of genus zero. However, in the present case they are not symmetric, instead the involution maps each of these Riemann surfaces onto the other. The real locus consists only of the two points $\pm r$. This can easily be understood by observing that the value $\alpha = -(r^2+1)^2$ is the global infimum of $Q(z,\bar{z})$.

In both of the reducible cases Bezout's Theorem says that there should be four points of intersection between the two curves (since these have degree two). These intersection points are the two points of infinity $(0:1:0)$ and $(0:0:1)$ plus, in the first case the intersection points of the two circles (in the real), and in the second case the two points $\pm r$.

Besides the above two special values of α, also the quadrature value $\alpha = 0$ is exceptional. This is a local maximum value for $Q(z, \bar{z})$. The local maximum is attained at $z = 0$, and the corresponding point $(0, 0)$ on the algebraic curve is a singular point (since both partial derivatives of Q vanish there). Thus $e = 1$ in the genus formula (26), hence $p = 0$.

We now embark on the full classification. Pictures for the case $r = \sqrt{2}$ with $\alpha = 3, -0.5, -1, -1.5$ are shown in Figures 1 and 2, where the shaded areas are the sets where $Q_\alpha(z, \bar{z}) < 0$.

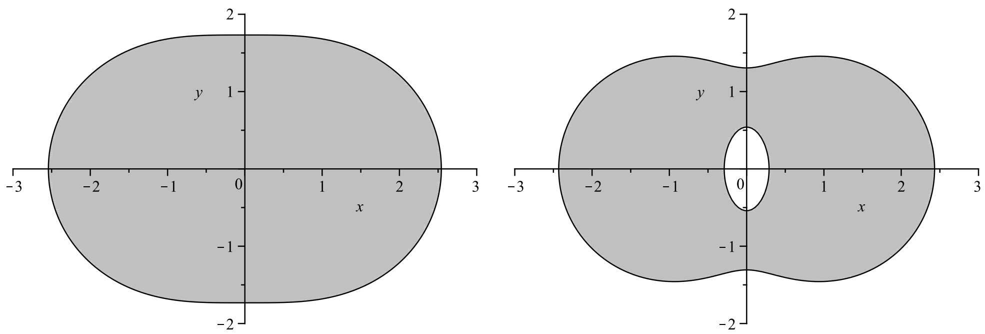

FIGURE 1. The level set Γ_α for $r = \sqrt{2}$ and for $\alpha = 3$ and $\alpha = -0.5$

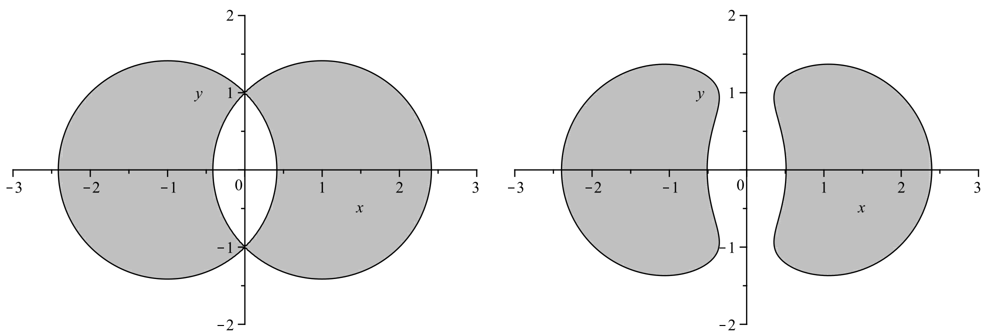

FIGURE 2. The level set Γ_α for $\alpha = -1$ and $\alpha = -1.5$

- For $\alpha > 0$, Γ_α has one component and there are no singular points visible in the real. Therefore, by (26) and the remark following it M_α has genus one. This means that the symmetry line Γ_α is not able to separate M_α into two halves (see more detailed discussions in [33, Sec. 2.2]). Thus $M_\alpha \setminus \Gamma_\alpha$ has only one component, and it will not generate any algebraic domain (even multi-sheeted), despite the nice picture in the real, with a smooth algebraic curve bounding a simply connected region (Figure 1, left). M_α can be viewed as the double of a Möbius band.
- At $\alpha = 0$ the genus of M_α collapses to zero, Γ_α has still only one component even though $\operatorname{loc}_{\mathbb{R}} Q_\alpha$ has acquired an additional point, $(0, 0)$, which is a singular point of $\operatorname{loc} Q_\alpha$ (resolved on M_α). It follows that $M_\alpha \setminus \Gamma_\alpha$ has two components, one of which, say M_+, is mapped conformally onto the quadrature domain $[\Omega]$ by the analytic function f which corresponds to the projection $(z, w) \mapsto z$ on $\operatorname{loc} Q_\alpha$.
- For $-(r^2 - 1)^2 < \alpha < 0$ the genus of M_α is again one, and the singular point $(0, 0)$ in the previous case has now grown to a curve, hence Γ_α has two components. Also $M_\alpha \setminus \Gamma_\alpha$ has two components, say $M_\pm$, and M_α can be viewed as the double of M_+ (or M_-), which topologically is an annulus.

  The meromorphic function $f\colon M_\alpha \to \mathbb{P}$ which corresponds to $(z, w) \mapsto z$ on $\operatorname{loc} Q_\alpha$ is however no longer univalent (not even locally univalent) on what corresponds to M_+ in the previous case. Therefore this gives now an only multi-sheeted algebraic domain, with $f(M_+)$ consisting of a main piece which contains the points ± 1 and the origin (the dashed area in Figure 1, right) plus a smaller piece around the origin (the bounded undashed region). The latter piece thus is covered twice, and the two sheets are connected via two branch points of the Schwarz function.
- For $\alpha = -(r^2 - 1)^2$ the curve is reducible, hence M_α is the union of two independent Riemann surfaces, both of genus zero and symmetric under J_α. In the real locus $\operatorname{loc}_{\mathbb{R}} Q_\alpha$ we simply have two intersecting circles, those centered at ± 1 and having radius r (Figure 2, left). Explicitly:

$$Q_\alpha(z, \bar{z}) = \left(|z - 1|^2 - r^2\right)\left(|z + 1|^2 - r^2\right).$$

  M_α is the union of two Riemann spheres and can be viewed as the double of two disks.
- For $-(r^2 + 1)^2 < \alpha < -(r^2 - 1)^2$ we are back to the case of genus one with Γ_α and $M_\alpha \setminus \Gamma_\alpha$ both having two components. However, the situation has changed in the sense that the involution goes the other way (like $z \mapsto -\bar{z}$ in place of $z \mapsto \bar{z}$ in a right-angled period parallelogram), it may be more natural in this case to think of M_α as the double of a cylinder than as the double of an annulus (even though these two types of domains are topologically equivalent). The real locus consists of two closed curves, one enclosing two branch points close to $z = 1$, the other enclosing two branch points close to $z = -1$, and with $f\colon M_\alpha \to \mathbb{P}$ as before, f maps M_+ (say)

onto the dashed region to the right in Figure 2 (right) covered twice and the (unbounded) undashed region covered once.
- When $\alpha = -(r^2+1)^2$ the two closed curves in the real locus of the previous case have shrunk to two points, the minimum points $z = \pm 1$ of $Q(z, \bar{z})$. The curve is reducible and M_α hence is the disjoint union of two Riemann surfaces (of genus zero), but these are not symmetric under J. Instead the involution J maps each of them onto the other. The algebraic curve $\mathrm{loc}_{\mathbb{R}}\, Q_\alpha$ consists of two pieces, which meet each other in two points of tangency. This is what is seen in the real locus. Thus M_α can be viewed as the double of the Riemann sphere, and J_α has no fixed points.
- When $\alpha < -(r^2+1)^2$ finally, there is no real locus at all. The genus is one, but now J has no fixed points at all (Γ_α is empty). Therefore no algebraic domain (even multi-sheeted) can be associated to this case. M_α can be viewed as the double of a Klein's bottle (see again [2], and also [33] for doubles of non-orientable surfaces).

The above can be compared with related analysis of spectral curves in, for example, [22]. We summarize the discussion in the following proposition, in which the last assertion is based on the general structure (20) of the elimination function.

Proposition 3. *The surface M_α (possibly disconnected) can be viewed as the double of the Klein surface $N_\alpha = M_\alpha/J_\alpha$, which in the different regimes of $\alpha \in \mathbb{R}$ is of the following topological type.*

- *For $\alpha > 0$: a Möbius band*
- *For $\alpha = 0$: a disk.*
- *For $-(r^2-1) < \alpha < 0$: an annulus.*
- *For $\alpha = -(r^2-1)^2$: two disjoint disks.*
- *For $-(r^2+1)^2 < \alpha < -(r^2-1)^2$: a cylinder.*
- *For $\alpha = -(r^2+1)^2$: a sphere.*
- *For $\alpha < -(r^2+1)^2$: a Klein's bottle.*

The pair (M_+, f) defines a single-sheeted algebraic domain for $\alpha = 0$ and a multi-sheeted algebraic domain for $-(r^2-1)^2 < \alpha < 0$ and $-(r^2+1)^2 < \alpha < -(r^2-1)^2$. For $\alpha = -(r^2-1)^2$ it defines two single-sheeted algebraic domains (which intersect in the complex plane).

The exponential transform of the above multi-sheeted algebraic domains is given by Theorem 2, where the elimination function is

$$\mathcal{E}_{f,f^*}(z, \bar{w}; a, \bar{b}) = \frac{Q_\alpha(z, \bar{w})Q_\alpha(a, \bar{b})}{Q_\alpha(z, \bar{b})Q_\alpha(a, \bar{w})}$$

with Q_α as in (24) *and the Schwarz function is given by* (25).

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