External geometry of p-minimal surfaces

Vladimir G. Tkachev *

Abstract. A surface $\mathcal{M}$ is said to be p-minimal if one of the coordinate functions is p-harmonic in the inner metric. We show that in the two dimensional case the Gaussian map of such surfaces is quasiconformal. In the case when the surface is a tube we study the geometrical structure of such surfaces. In particular, we establish the second order differential inequality for the form of the sections of $\mathcal{M}$ which generalizes the known ones in the theory of minimal surfaces.

1. Introduction

1.1. Let $\mathcal{M} = (M; x)$ be a surface given by a C^2-immersion $x : M \to \mathbb{R}^{n+1}$ of an n-dimensional orientable noncompact manifold M.

Definition 1. A surface $\mathcal{M}$ is said to be *minimal* if its mean curvature vector $H(m) \equiv 0$.

The well-known property of minimal immersions in Euclidean space is the harmonicity of their coordinate functions. Moreover, if *one* coordinate function of an immersion is harmonic then *all* coordinates satisfy this property and the immersion is minimal. On the other hand, for $n = 2$ this condition yields the fact that the Gauss map of such surfaces is conformal [14].

The natural question arises: what happens if we replace the requirement of harmonicity by p-harmonicity?

Definition 2. For a fixed $p > 1$ a surface $\mathcal{M}$ is said to be *p-minimal* if one of the coordinate functions is p-harmonic with respect to the inner metric of $\mathcal{M}$. In other words, there exists a direction $e \in \mathbb{R}^n$ such that

$$\Delta_p f \equiv \operatorname{div} |\nabla f|^{p-2} \nabla f = 0, \qquad (1.1)$$

where $f(m) = \langle x(m), e \rangle$ and ∇ is the covariant derivative on M.

One can easily show that p-harmonicity of one coordinate function can't be extended to the others provided $p \neq 2$. This means that $\mathbb{R}^{n+1}$ is equipped with a specified

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direction e. This kind of asymmetry is typical for the Minkowski spaces $\mathbb{R}_1^{n+1}$ with time-axis Oe. Another example is the tubular minimal surfaces (see the definition below) which are Euclidean analogues of the relative strings in nuclear physics.

We should also mention that equation (1.1) is of great importance in the nonlinear potential theory [2] and the elliptic type PDE's [7], [4]. On the other hand, it is closely linked to quasiregular mappings (see [3] for detailed information).

This paper is devoted to two sides of the p-minimal surface theory only. The first part of our paper concerns the basic facts of p-minimal surface theory itself. In particar, we show that the Gauss map of a two-dimensional p-minimal surface is $K(p)$-quasiconformal. In Section 3 we apply a new method to study the shape of p-minimal tubes. In particular, we establish the estimates for the sizes of sections of such surfaces which give us information about the evolution of a tube in the time-direction.

It is the aim of this paper to demonstrate the properties which are common to the tubular minimal and p-minimal surfaces. We only touch on the nonparametric case for p-minimal surfaces and don't mention the properly immersed non-tubular p-minimal surfaces at all. The general theory of p-minimal surfaces and some examples will be given in a paper in preparation.

2. Preliminary properties of p-minimal surfaces

2.1. We have noticed above that the case $p = 2$ corresponds to minimal surfaces. To clarify the geometrical meaning of (1.1) for arbitrary p we denote by $k_e(m)$ the curvature of $\mathcal{M}$ in the e-direction (*i.e.* the sectional curvature of the 2-plane spanned by e and the unit normal ν to $\mathcal{M}$ at a point m).

Proposition 1. *Let m be a noncritical point, i.e., $e \wedge \nu(m) \neq 0$. Then*

$$H(m) = -(p-2)k_e(m). \tag{2.2}$$

Proof. Really, let $\overline{\nabla}$ and ∇ denote the standard covariant derivatives in $\mathbb{R}^{n+1}$ and $\mathcal{M}$ respectively. Then

$$\nabla f(m) = \left(\overline{\nabla}\langle x(m), e\rangle\right)^{\top} = e^{\top},$$

where $e^{\top}$ is the projection of e onto the tangent space to the surface $\mathcal{M}$ at a point m. It follows from the assumptions of the proposition that $|e^{\top}| \neq 0$ or, equivalently, $|\nabla f(m)| \neq 0$. Thus, for any tangent vector X we obtain

$$\nabla_X|\nabla f| = \nabla_X|e^{\top}| = \frac{\langle\overline{\nabla}_X e^{\top}, e^{\top}\rangle}{|e^{\top}|} = \frac{\langle\overline{\nabla}_X(-e^{\perp}), e^{\top}\rangle}{|e^{\top}|} = \langle e, \nu\rangle\langle A(X), \frac{e^{\top}}{|e^{\top}|}\rangle.$$

Here A is the Weingarten map of $\mathcal{M}$ and $e^{\perp}$ is the projection of e onto the normal space to $\mathcal{M}$. By virtue of the symmetry of A we conclude that

$$\nabla|\nabla f| = \langle e, \nu\rangle A(\tau), \tag{2.3}$$

where $\tau = e^{\top}/|e^{\top}|$ is well defined at m. After substituting (2.3) into (1.1) we have

$$\begin{aligned}\Delta_p f &= \operatorname{div}(|\nabla f|^{p-2}\nabla f) = (p-2)|\nabla f|^{p-3}\langle\nabla f, \nabla(|\nabla f|)\rangle + |\nabla f|^{p-2}\Delta f \\ &= |\nabla f|^{p-4}\left(|e^{\top}|^2\Delta f + \langle e, \nu\rangle(p-2)\langle A(e^{\top}), e^{\top}\rangle\right).\end{aligned} \tag{2.4}$$

Finally, the definition of $k_e(m)$ together with the well known connection between the mean curvature $H(m)$ and the inner Laplacian [6]: $\Delta f(m) = H(m)\langle e, \nu\rangle$ yield from (2.4)

$$\Delta_p f = |\nabla f|^{p-4}\langle e, \nu\rangle|e^{\top}|^2\big(H(m) + (p-2)k_e(m)\big), \tag{2.5}$$

everywhere in the regular part $M_0 \equiv \{m \in M : |e^{\top}(m)| \neq 0\}$.

We now assume that equality (2.2) doesn't hold at some noncritical point $m_1 \in M_0$. Then in view of (2.5) and (1.1), $\langle e, \nu(m_1)\rangle \equiv 0$, and by continuity of the expression in parentheses in (2.5), the last identity holds everywhere in some neighbourhood $\Omega(m_1)$. Thus, in $\Omega(m_1) \cap M_0$ the coordinate function $f(m) = \langle e, x\rangle$ is constant and, it follows that $A \equiv 0$ in $\Omega(m_1)$. But this conclusion trivially yields (2.2), which contradicts our assumption. □

The following assertion is an immediate consequence of the Meusnier theorem.

Corollary 1. *The mean curvature H of a p-minimal surface $\mathcal{M}$ and the mean curvature h of the section $\Sigma(\tau)$ are related by*

$$h(m) = -\frac{p-1}{\omega}k_e(m) = \frac{p-1}{p-2}\frac{H(m)}{\omega} \tag{2.6}$$

where $\omega = \langle \nu_m, e\rangle$.

We use further the auxiliary assertion which clarifies the local structure of a p-minimal surface near a critical point. We notice that this property has no analogue in minimal surface theory.

Lemma 1. *Let $\mathcal{M}$ be a p-minimal surface given as a graph of a C^2-function $f(x)$ defined in a domain $G \subset \mathbb{R}^n$. Let $x_0 \in G$ be a critical point of $f(x)$, i.e. $\overline{\nabla} f(x_0) = 0$. Then the Hessian $\overline{\nabla}^2 f$ is degenerate. In other words, x_0 is a planar point.*

Proof. To prove this assertion we rewrite (2.2) in a more suitable way. In the local coordinates we have the following formulas for the mean curvature $H(m)$ and the Laplace-Beltrami operator Δ respectively:

$$\begin{aligned} H(m) &= \frac{1}{g^{3/2}}\sum_{i,j=1}^{n}(g\delta_{ij} - \overline{\nabla}_i f\overline{\nabla}_j f)\overline{\nabla}^2_{ij} f, \\ \Delta u &= \frac{1}{\sqrt{g}}\sum_{i,j=1}^{n}\overline{\nabla}_j(g^{ij}\sqrt{g}\,\overline{\nabla}_j u),\end{aligned} \tag{2.7}$$

where $\overline{\nabla}_i$ denotes the covariant derivative along the coordinate vector e_i, g^{ij} is the inverse matrix to the metric tensor $g_{ij} = \delta_{ij} + \overline{\nabla}_i f \overline{\nabla}_j f$ and $g = \det \|g_{ij}\|$. Hence, we obtain from (1.1) and (2.4)

$$g|\overline{\nabla} f|^2 \operatorname{tr} \overline{\nabla}^2 f + \sum_{l,s=1}^{n} (p - 2 - |\overline{\nabla} f|^2) \overline{\nabla}_l f \overline{\nabla}_s f \overline{\nabla}^2_{ls} f = 0, \tag{2.8}$$

Here $\overline{\nabla}^2 f$ is the Hessian of $f(x)$ and the trace $\operatorname{tr} \overline{\nabla}^2 f$ is equal to the euclidean Laplace operator in $\mathbb{R}^n$. Write $a_{ij} = \overline{\nabla}^2_{ij}(x_0)$ and $A = ||a_{ij}||$. Then for an appropriate choice of $\varepsilon > 0$ and every vector $y \in \mathbb{R}^n$ such that $|y| < \varepsilon$ we have

$$\overline{\nabla}_k f(x_0 + y) = \sum_{i=1}^{n} a_{ki} y_i + o(|y|),$$

and

$$|\overline{\nabla} f(x_0 + y)|^2 = O(|y|^2).$$

Substituting these relations in (2.8), we arrive at

$$\sum_{k,l,s=1}^{n} \sum_{i,j=1}^{n} (a_{ki} a_{kj} \operatorname{tr} A + (p - 2) a_{li} a_{sj} a_{ls}) y_i y_j = o(|y|^2).$$

Taking into account the validity of the last equality for all sufficiently small $y \in \mathbb{R}^n$ we obtain a matrix equation

$$A^2(I \operatorname{tr} A + (p - 2)A) = 0, \tag{2.9}$$

where I is the unit matrix. By virtue of the symmetry of the Hessian A we can choose an orthonormal basis of $\mathbb{R}^n$ consisting of the eigenvectors of A. Namely, A takes a diaganal form $\lambda_i \delta_{ij}$ and from (2.9) we have for $i : 1 \leq i \leq n$,

$$\lambda_i \big(\lambda_i (p - 2) + \operatorname{tr} A\big) = 0.$$

We see from the last identity that all non-zero eigenvalues λ_i must be equal to $-(p - 2)^{-1} \operatorname{tr} A$. Let $\lambda_1, \dots, \lambda_k$ be all such numbers. Then after summing we obtain

$$\operatorname{tr} A = \sum_{i=1}^{k} \lambda_i = -\frac{k}{p - 2} \operatorname{tr} A. \tag{2.10}$$

On the other hand $\operatorname{tr} A = k\lambda_1 \neq 0$. It follows from (2.10) that $p = 2 - k$, where $k \geq 1$ is a positive integer. But this contradicts $p > 1$ and hence, all λ_i are zero. Now the theorem follows from the standard properties of symmetric matrices. □

2.2. Given a surface $\mathcal{M}$ in $\mathbb{R}^3$ we denote by $\gamma(m) : \mathcal{M} \to S^2$ the standard Gauss map. A result of Gauss states that, if the surface is minimal, that map is conformal. Here we extend this property to p-minimal surfaces. First we recall

Definition 3 ([1], [2]). A map $F : M_1 \to M_2$ of two smooth Riemannian manifolds M_1, M_2 is called a *quasiconformal map* if the Jacobian $\det d_x F$ doesn't change sign on M_1 and for almost every $x \in M_1$,

$$\max |d_x F(E)| \le K_m \min |d_x F(E)| \tag{2.11}$$

where min and max are given over all unit tangent vectors E of $T_x M_1$. The number $K = \max_{m \in M_1} K_m$ is called the *distortion coefficient* of F.

Theorem 1. *Let $\mathcal{M}$ be a two dimensional p-minimal surface in $\mathbb{R}^3$. Then the Gauss map is a $K(p)$-quasiconformal map with distortion coefficient*

$$K(p) = \max\{p-1; 1/(p-1)\}. \tag{2.12}$$

Proof. We notice that the tangent spaces $T_m M$ to M and $T_{\gamma(m)} S^2$ to the unit sphere S^2 can be regarded as canonical isomorphic ones. Really, we identify the vector $A(E)$ with $d\gamma_m(E)$, where $d\gamma_m$ is the differential of the Gauss map at m. We specify a point $m \in M$ and choose an orthonormal basis E_1, E_2 of the tangent space $T_m M$ which diagonalizes A, *i.e.*

$$A(E_i) = \lambda_i E_i,$$

where λ_1, λ_2 are the principal curvatures of $\mathcal{M}$ at m. Without loss of generality we can arrange that $|e^T(m)| \neq 0$. Really Lemma 1 yields that the homomorphism A is identically zero and (2.11) is trivial.

Let us denote $\tau = e^T/|e^T|$. Then for some angle $\psi \in [0; 2\pi]$,

$$\tau = E_1 \cos\psi + E_2 \sin\psi,$$

and by the Meusnier theorem we have

$$\langle A\tau, \tau\rangle = \lambda_1 \cos^2\psi + \lambda_2 \sin^2\psi = -\frac{1}{p-2}(\lambda_1 + \lambda_2).$$

Hence

$$\lambda_1 = -\lambda_2 \frac{1 + (p-2)\sin^2\psi}{1 + (p-2)\cos^2\psi}.$$

It is a direct consequence of the last identity that the Jacobian $\det(d_m\gamma) = \lambda_1\lambda_2$ must be negative. Standard facts of quadratic form theory allows us to conclude that the distortion coefficient of γ at a point m is less or equal to

$$K_m = \max_{\psi} \left\{q; \frac{1}{q}\right\}, \qquad q = \frac{1 + (p-2)\sin^2\psi}{1 + (p-2)\cos^2\psi}.$$

Then varying ψ we obtain the required maximum of K_m. □

L. Simon in [16] established that every entire two dimensional nonparametric surface with quasiconformal Gauss map must be a plane. As a consequence of this result we obtain a version of the well-known Bernstein theorem.

Corollary 2. *Let $\mathcal{M}$ be an entire p-minimal graph in $\mathbb{R}^3$. Then $\mathcal{M}$ is a plane.*

Remark. It follows from [10] and [15] that every minimal n-dimensional graph $\mathcal{M}$ in $\mathbb{R}^{n+1}$ has parabolic conformal type. In other words, every compact set on $\mathcal{M}$ has zero n-capacity. In these papers we applied the quasiconformal mapping theory to minimal surfaces. The methods used there allows us to conclude that a similar property holds for p-minimal graphs also if $p \geq n$. These facts together with Corollary 2 make the following very plausible:

Conjecture. Let $\mathcal{M}$ be an entire p-minimal graph given over the whole of $\mathbb{R}^n$. If $p \geq n$ then $\mathcal{M}$ is a hyperplane.

3. Tubular p-minimal hypersurfaces

3.1. In this section we deal with tubular type p-minimal surfaces. This class of surfaces in the two dimensional case was investigated by J. C. C. Nitsche [13] and have been studied by V. M. Miklyukov [9] in the high dimensional situations.

Definition 4. We say that a surface $\mathcal{M}$ is a *tube* with the projection interval $\tau(\mathcal{M}) \subset Ox_{n+1}$, if

(1) for any $\tau \in \tau(\mathcal{M})$ the sections $\Sigma_\tau = x(M) \cap \Pi_\tau$ by hyperplanes $\Pi_\tau = \{x \in \mathbb{R}_1^{n+1} : x_{n+1} = \tau\}$ are not empty compact sets;

(2) for $\tau', \tau'' \in \tau(\mathcal{M})$ any part of $\mathcal{M}$ situated between two different $\Pi_{\tau'}$ and $\Pi_{\tau''}$ is a compact set.

If $\tau(\mathcal{M})$ is an infinite interval we call the surface an *infinite tube*. Otherwise, we call a length of $\tau(\mathcal{M})$ the *life-time* of $\mathcal{M}$.

Let

$$\rho(\tau) = \max_{m \in \Sigma(\tau)} \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}.$$

It follows from the results of [12], [11], [5], that every n-dimensional minimal tube of arbitrary codimension satisfies the following differential inequality

$$\rho(\tau)\rho''(\tau) \geq (n-1)(1 + \rho'(\tau)^2), \tag{3.13}$$

which is crucial for the theory of minimal tubes. As a consequence every minimal tube for $n \geq 3$ is contained in a slab between two parallel planes. Hence, there are no many-dimensional infinite minimal tubes. In contrast, the two dimensional case essentially differs from the high dimensional one: there are tubes of finite life-time as well as infinite tubes. Moreover, we show in [19] that the life-time in the first case is derived by the angle between the full-flow vector of a minimal tube and the time-axis.

Lemma 2. *Let V be a convex compact set in $\mathbb{R}^n$ and W be a compact set such that $W \setminus V \neq \emptyset$. Then there exists a closed ball $B \subset \mathbb{R}^n$ such that*

$$W \subset B \tag{3.14}$$

and

$$\partial B \cap (W \setminus V) \neq \emptyset. \tag{3.15}$$

Proof. The distance function $f(x) = \operatorname{dist}(x, V)$ is continuous on $\mathbb{R}^n$. It follows from the conditions of the lemma that this function attains the maximum value on W at some point $a \in W$ and $d = f(a) > 0$. On the other hand, by virtue of the convexity of V there exists a unique point $b \in \partial V$ such that $f(a) = \|b - a\|$.

Choose a new coordinate system of $\mathbb{R}^n$ with the origin at a: let the first coordinate vector be

$$e_1 = \frac{b-a}{d},$$

and choose the others, $e_2, \ldots, e_n$ so that we have an orthonormal system. Then, the hyperplane given by $x_1 = d$ is one of support to V at a. It follows from the triangle inequality that W is contained in a halfspace $\{x_1 \geq 0\}$ and V in $\{x_1 \geq d\}$.

Given positive h and R we specify an open ball

$$B(R, h) = \{x \in \mathbb{R}^n : (x_1 + R)^2 + x_2^2 + \cdots + x_n^2 < (R + h)^2\}.$$

By our choice and the compactness of V, given a positive ϵ there exists $R > 0$ such that V is contained in a ball $B(R, \epsilon)$.

Suppose $\epsilon = d/2$ and R_0 is the corresponding radius. Then the definition of d yields that $a \notin B(R_0, d/2)$, however the greater ball $B(R_0, 3d/2)$ contains V as well as W. Let δ_0 be the minimum over all $\delta \in (0; d)$ such that

$$W \subset \overline{B(R_0, d/2 + \delta)}.$$

Then $a \in \partial B$, where $B = \overline{B(R_0, d/2 + \delta_0)}$ and $V \cap B = \emptyset$. □

Corollary 3 (Maximum Principle). *Let $\mathcal{M} = (M, x)$ be an immersed compact p-minimal hypersurface in $\mathbb{R}^{n+1}$ with nonempty boundary ∂M. Then*

$$\operatorname{conv} x(\partial M) = \operatorname{conv} x(M), \tag{3.16}$$

where conv E *is the convex hull of* E.

Proof. Let us denote $\Omega = \operatorname{conv} x(\partial M)$ and assume that (3.16) fails. Then this implies $x(M) \setminus \Omega \neq \emptyset$. By Lemma 2 we can find a closed ball B such that $x(M) \subset B$ and there exists a point $m \in \operatorname{int} M$, $x(m) \in \partial B$. We choose a neighbourhood $\mathcal{O}$ of m such that the restriction of x on $\mathcal{O}$ is an embedding. Further arguments will be local and we can arrange that $\mathcal{M} = x(\mathcal{O})$ without loss of generality.

Because of the choice of B, the tangent spaces to $\mathcal{M}$ and ∂B at $x(m)$ coincide. Moreover, $\mathcal{M} \subset B$ and the standard comparison principle for touching surfaces gives the following inequality

$$\lambda_i \geq \frac{1}{R}, \tag{3.17}$$

where λ_i are the principal curvatures of $\mathcal{M}$ at m with respect to the inward normal of ∂B and R is the radius of B.

We now turn to identity (2.2). By the definition of $k_e(m)$ there exists a system of positive numbers $\alpha_i \leq 1$ such that

$$\sum_{i=1}^{n} \alpha_i = 1 \qquad \text{and} \qquad \mathrm{k_e(m)} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \alpha_\mathrm{i}\lambda_\mathrm{i}.$$

It follows from these relations, (3.17) and (2.2), that

$$0 = \sum_{i=1}^{n} \lambda_i(1 + (p-2)\alpha_i) \geq \frac{n+p-2}{R} > 0.$$

This contradiction proves Corollary 3. □

3.2. Further, we use the Minkowski operations. Namely, given $A, B \subset \mathbb{R}^n$ the notations $A \oplus B$ and λA denote the sets $\{x = a + b : a \in A, b \in B\}$ and $\{x = \lambda a : a \in A\}$.

Definition 5. A family of convex sets $\{\Omega(\tau) : \tau \in [\alpha, \beta]\}$ is called [8] *convex* if for arbitrary $\tau_1 < \tau_2$ from interval $[\alpha; \beta]$ and a nonnegative $t \leq 1$ one has

$$\Omega\,(\tau_1 t + \tau_2(1-t)) \subset t\Omega(\tau_1) \oplus \bar{t}\Omega(\tau_2).$$

Let $\mathcal{M}$ be an n-dimensional p-minimal tube in $\mathbb{R}^{n+1}$. Let us denote by $\Omega(\tau)$ the projection of the convex hull of the section $\Sigma(\tau)$ onto the hyperplane $\Pi_0 = \{x_{n+1} = 0\}$. Then

$$\operatorname{conv}\Sigma(\tau) = \tau e_{n+1} \oplus \Omega(\tau).$$

Theorem 2. *The family* $\{\Omega(\tau) : \tau \in \tau(\mathcal{M})\}$ *is convex.*

Proof. We specify $\tau_1 < \tau_2$ from the interval $\tau(\mathcal{M})$ and $t \in [0; 1]$. Let H be the slab $\{x : x_{n+1} \in (\tau_1; \tau_2)\}$ and $M' = x^{-1}(H \cap x(M))$. Then Corollary 3 gives

$$V \equiv \operatorname{conv}(\Sigma(\tau_1) \cup \Sigma(\tau_2)) = \operatorname{conv} x(M').$$

Let $\tau_0 = t\tau_1 + \bar{t}\tau_2$. Then $\Sigma(\tau_0) \subset V$ and by the definition of the convex hull we conclude that $\operatorname{conv}\Sigma(\tau_0) \subset V$.

We choose an arbitrary $z \in \Omega(\tau_0)$. Then $y = z + \tau_0 e_{n+1} \in \Pi(\tau_0) \cap V$, and there exist $y_i \in \operatorname{conv} \Sigma(\tau_i)$ and $\lambda \in [0; 1]$ such that

$$y = \lambda y_1 + \overline{\lambda} y_2. \tag{3.18}$$

Then the decomposition of $y_i = z_i + \tau_i e_{n+1}$ for certain $z_i \in \Omega(\tau_i)$ and (3.18) gives

$$z = \lambda z_1 + \overline{\lambda} z_2, \qquad \tau_0 = \lambda \tau_1 + \overline{\lambda} \tau_2.$$

Hence, $\lambda = t$ and it follows that $z \in t\Omega(\tau_1) \oplus \overline{t}\Omega(\tau_2)$ as required. □

The following assertion gives a sample of applications of the last result.

Corollary 4. *Let $R(\tau)$ be the radius of the least ball which contains $\Sigma(\tau)$* (such a ball is said to be circumscribed near $\Sigma(\tau)$). *Then $R(\tau)$ is a convex function.*

Proof. We denote by $B(\tau)$ the projection onto Π_0 of the ball circumscribed near $\Sigma(\tau)$. Then, by virtue of convexity of $B(\tau)$ we have $B(\tau) \supset \Omega(\tau)$, and Theorem 2 yields for arbitrary $t \in [0; 1]$:

$$\Omega(\tau_0) \subset t\Omega(\tau_1) \oplus \overline{t}\Omega(\tau_2) \subset tB(\tau_1) \oplus \overline{t}B(\tau_2) = B_0,$$

where $\tau_0 = \tau_1 t + \tau_2(1-t)$. By definition, $R(\tau_0) \le R_0$, where R_0 is the radius of B_0. On the other hand, $R_0 = tR(\tau_1) + \overline{t}R(\tau_2)$ and we obtain the required inequality

$$R(\tau_1 t + \tau_2(1-t)) \le R(\tau_1) + \overline{t}R(\tau_2).$$

□

3.3. Now we study the structure of $\Sigma(\tau)$ more completely. This requires further delicate information not only about $R(\tau)$ but about the curve of the centers of the balls $B(\tau)$ as well. Let us denote by $\xi(\tau)$ the center of $B(\tau)$. We recall without proof the well known extremal property of $B(\tau)$ (see [8], Theorem 7.5).

Lemma 3. *Let E be a closed subset of $\mathbb{R}^n$ and $B(E)$ the ball circumscribed near E with the center ξ. Then for all unit vectors $y \in \mathbb{R}^n$ there exists $b \in \partial E \cap \partial B(E)$ such that*

$$\langle b - \xi, y \rangle \ge 0. \tag{3.19}$$

We denote by

$$\sigma(E) = \min_{y \in S^{n-1}} \max_{b \in \partial B \cap E} \frac{\langle b - \xi, y \rangle}{R},$$

where B is the circumscribed ball near a compact set E, R is the radius and ξ is the center of B. It follows from (3.19) that $0 \le \sigma(E) \le 1$. Moreover, one easily shows that $\sigma(E) = 0$ if and only if the intersection of the boundary sphere $S = \partial B$ with F lies in some equatorial hemisphere of S.

Theorem 3. *Let $\mathcal{M}$ be a p-minimal tube in $\mathbb{R}^{n+1}$ such that*

$$\sigma(\Sigma(\tau)) \geq \epsilon > 0, \qquad \forall \tau \in \tau(\mathcal{M}). \tag{3.20}$$

Then $\xi(\tau)$ is a δ-convex curve of τ. In other words, any coordinate function $\xi_k(\tau)$ admits the composition

$$\xi_k(\tau) = \varphi_k(\tau) - \psi_k(\tau),$$

with $\varphi_k(\tau)$, $\psi_k(\tau)$ being convex functions.

Proof. We consider τ_1, τ_2 from $\tau(\mathcal{M})$ and $t \in [0; 1]$. Let us denote by $B(\tau_i) = B_i(\xi(\tau_i), R_i)$ the corresponding balls circumscribed near $\Sigma(\tau_i)$. As above we have for $\tau_0 = t\tau_1 + \bar{t}\tau_2$

$$\Omega(\tau_0) \subset tB(\tau_1) \oplus \bar{t}B(\tau_2).$$

By Lemma 3 we can find $y \in \partial B(\tau_0) \cap \Sigma(\tau_0)$ such that

$$\langle y - \xi(\tau_0), \xi(\tau_0) - \xi_0 \rangle \geq \epsilon |y - \xi(\tau_0)| \cdot |\xi(\tau_0) - \xi_0|,$$

where $\xi_0 = t\xi(\tau_1) + \bar{t}\xi(\tau_2)$. Hence,

$$\begin{aligned} |y - \xi_0|^2 &= \left|(y - \xi(\tau_0)) + (\xi(\tau_0) - \xi_0)\right|^2 \\ &\geq |y - \xi(\tau_0)|^2 + |\xi(\tau_0) - \xi_0|^2 + 2\epsilon |y - \xi(\tau_0)| \cdot |\xi(\tau_0) - \xi_0|, \end{aligned}$$

and taking into account that $|y - \xi(\tau_0)| = R(\tau_0)$ and $|y - \xi_0| \leq R_0$ we obtain

$$|\xi(\tau_0) - \xi_0|^2 + 2\epsilon |y - \xi(\tau_0)| \cdot |\xi(\tau_0) - \xi_0| + (R^2(\tau_0) - R_0^2) \leq 0,$$

and as a consequence,

$$|\xi(\tau_0) - \xi_0| \leq \frac{R_0^2 - R^2(\tau_0)}{R(\tau_0)\epsilon + \sqrt{R_0^2 - R^2(\tau_0)(1 - \epsilon^2)}}. \tag{3.21}$$

By Corollary 4 we have $R_0 \geq R(\tau_0)$ and from (3.21),

$$|\xi(\tau_0) - \xi_0| \leq \frac{R_0^2 - R^2(\tau_0)}{\epsilon(R(\tau_0) + R_0)} = \frac{1}{\epsilon}(R_0 - R(\tau_0)). \tag{3.22}$$

We consider the coordinate function $\xi_k(\tau) = \langle \xi(\tau), e_k \rangle$. Then (3.22) yields

$$t\xi_k(\tau_1) + \bar{t}\xi_k(\tau_2) - \xi_k(\tau_0) \leq \frac{1}{\epsilon}\big(tR(\tau_1) + \bar{t}R(\tau_2) - R(\tau_0)\big).$$

This inequality means that the difference $\psi(\tau) = \epsilon^{-1}R(\tau) - \xi_k(\tau)$ is convex. Therefore, by Corollary 4 we obtain the required decomposition of $\xi_k(\tau)$ into the difference of two convex functions

$$\xi_k(\tau) = \frac{1}{\epsilon}R(\tau) - \psi(\tau),$$

and the lemma is proved. □

Theorem 4. *Let $\mathcal{M}$ be a p-minimal surface with assumption* (3.20) *and* $\beta = (n-1)/(p-1)$. *Then $R(\tau)$ and $\xi(\tau)$ satisfy the differential inequality*

$$R(\tau)R''(\tau) \geq \beta(1+R'(\tau)^2) + |\xi'(\tau)|^2 \min\{\beta; 1\} \tag{3.23}$$

almost everywhere in $\tau(\mathcal{M})$.

Proof. Convexity of a function provides existence a.e. of the second differential (see [8] or [2], Theorem 5.3). It follows from Corollary 4, Theorem 3 that $R(\tau)$ as well as $\xi_k(\tau)$ have the second differentials almost everywhere in $\tau(\mathcal{M})$. We denote by $\tau'(\mathcal{M})$ the set of full measure where the second differentials of $R(\tau)$ and $\xi_k(\tau)$, $1 \leq k \leq n+1$ do exist.

Let S^{n-1} be the unit sphere in $\Pi_0 \sim \mathbb{R}^n$ endowed with the standard metric. We consider the hypersurface $\mathcal{M}_0$ given by

$$w(\theta,\tau) = \xi(\tau) + R(\tau)\theta + \tau e_{n+1} : S^{n-1} \times \mathbb{R} \to \mathbb{R}^{n+1}$$

where $\theta \in S^{n-1}$. We have shown in [18] that for such a surface the curvature $k_{e,\mathcal{M}_0}$ in the e-direction can be calculated from

$$k_{e,\mathcal{M}_0}(\theta,\tau) = \frac{\omega^3}{R(\tau)}\left[R(\tau)R''(\tau) + R(\tau)\langle\xi''(\tau),\theta\rangle + \langle\xi'(\tau),\theta\rangle^2 - |\xi'|^2\right] \tag{3.24}$$

where

$$\omega^2 = \langle \nu_m, e\rangle^2 = \frac{1}{1+\left(R'(\tau)+\langle\theta,\xi'(\tau)\rangle\right)^2}.$$

By the definition of functions $R(\tau)$ and $\xi(\tau)$ we conclude that the surface $\mathcal{M}$ is contained in $\mathcal{M}_0$ in the sense that $\Sigma(\tau)$ is a subset of $\Pi(\tau) \cap \mathcal{M}_0$ for all $\tau \in \tau(\mathcal{M})$.

Let us consider an arbitrary $\tau \in \tau'(\mathcal{M})$ and $E = \Omega(\tau) \cap \partial B(\tau)$. The surfaces $\mathcal{M}$ and $\mathcal{M}_0$ have the common outward normal ν_m at $m = y \oplus \tau e_{n+1}$ for every $y \in E$ (we mean by *outward* the normal which is directed out from the inside of $B(\tau)$). Let $\mathcal{O}$ be the neighbourhood of m where $x(\cdot)$ is an embedding. It is a consequence of the definition of $\mathcal{M}_0$ that $\nu_m \wedge e_{n+1} \neq 0$. We denote by $\gamma(\tau)$ and $\gamma_0(\tau)$ the sections of $x(M)$ and $\mathcal{M}_0$ by the two-plane spanned by ν_m and e_{n+1}. Then the comparison principle for touching surfaces yields

$$k_{e,\mathcal{M}}(m) \leq k_{e,\mathcal{M}_0}(m).$$

We write $h(m)$ and $h_0(m)$ for the mean curvatures at m of the sections $\Sigma(\tau)$ and $\Pi(\tau) \cap \mathcal{M}_0 = \xi(\tau) \oplus \tau e_{n+1} \oplus B(\tau)$ with respect to their common outward normal. Then by the comparison principle we arrive at the inequality

$$h(m) \leq h_0(m) \equiv -\frac{n-1}{R(\tau)},$$

and after (2.6)

$$-\frac{p-1}{\omega} k_e(m) \leq -\frac{n-1}{R(\tau)}.$$

By (3.24) we obtain after simplification

$$R(\tau)R''(\tau) - \beta(1 + R'(\tau)^2) \geq (\beta - 1)\langle \xi'(\tau), \theta\rangle^2 + |\xi'|^2 + \langle \theta, y\rangle, \tag{3.25}$$

where $y = 2\beta R'(\tau)\xi'(\tau) - R(\tau\xi''(\tau))$. Thus, Lemma 3 applied to the vector y provides $b \in E$ such that $\langle b - \xi(\tau), y\rangle \geq 0$. We take

$$\theta_0 = \frac{b - \xi(\tau)}{R(\tau)}$$

and it follows from (3.25) that

$$R(\tau)R''(\tau) - \beta(1 + R'(\tau)^2) \geq (\beta - 1)\langle \xi'(\tau), \theta_0\rangle^2 + |\xi'(\tau)|^2 \geq |\xi'(\tau)|^2 \min\{\beta; 1\},$$

and the theorem is proved completely. □

Remark. Finally, we notice that the quantity $R(\tau)$ measures the size of the section $\Sigma(\tau)$ instead of the distance of this section from the time-axis in the previous inequalities (3.13). Moreover, in the base case $p = 2$ the established inequality (3.23) is stronger than (3.13).

On the other hand, δ-convex functions belong to the class $\overline{W}^2_{1,\mathrm{loc}}(\tau(\mathcal{M}))$; that is, they have a second-order generalized derivative that is a measure (see [2], Chapter 2, §4.10, Corollary). This allows us to proceed to the integration of (3.23) to completion in the standard way [12]:

Corollary 5. *Let $\mathcal{M}$ be a p-minimal tube,* $\dim \mathcal{M} = n > p > 1$. *Then it has finite life-time* length $\tau(\mathcal{M})$. *Moreover,*

$$\text{length}\, \tau(\mathcal{M}) \leq 2c_\beta r(\mathcal{M}), \quad \beta = \frac{n-1}{p-1}$$

where

$$r(\mathcal{M}) \equiv \min_{\tau \in \tau(\mathcal{M})} R(\tau) > 0$$

and

$$c_\beta = \int_0^{+\infty} \frac{dt}{(1 + t^{2\beta})^{1/2}}.$$

We notice that the previous inequality is precise for the p-minimal surfaces with rotational symmetry.

References

[1] L. Ahlfors, Lectures on quasiconformal mappings, Toronto–New York–London: Van Nostrand Math. Studies, 1966.

[2] V. M. Goldstein and Yu. G. Reshetnyak, Introduction to the theory of functions with generalized derivatives and quasiconformal mappings, Nauka, Moscow 1983.

[3] J. Heinonen, T. Kilpelainen and O. Martio, Nonlinear potential theory of degenerate elliptic equations, Oxford Univ. Press, London 1993.

[4] J. Heinonen, T. Kilpelainen and J. Rossi, The growth of A-subharmonic functions and quasiregular mappings along asymptotic paths, Indiana Univ. Math. J. 38 (1989), 581–601.

[5] V. A. Klyachin, Estimate of spread for minimal surfaces of arbitrary codimension, Sibirsk. Mat. Zh. 33 (5) (1992), 201–207.

[6] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Vol. 2, Interscience, 1969.

[7] P. Lindqvist, On the definition and properties of p-superharmonic functions, J. Reine Angew. Math. 365 (1986), 67–75.

[8] K. Leichtweiss, Konvexe Mengen, Springer-Verlag, 1980.

[9] V. M. Miklyukov, On some properties of tubular minimal surfaces in R^n, Dokl. Akad. Nauk SSSR 247 (1979), No. 3, 549–552; English transl. in Soviet Math. Dokl. 20 (1979).

[10] V. M. Miklyukov and V. G. Tkachev, On the structure in the large of externally complete minimal surfaces in $\mathbb{R}^3$, Soviet Math. (Iz. VUZ) 31 (1987), 30–36.

[11] V. M. Miklyukov and V. G. Tkachev, Some properties of tubular minimal surfaces of arbitrary codimension, Mat. Sb. 180 (9) (1989), 1278–1295; English transl. in Math. USSR Sb. 68 (1) (1991), 133–150.

[12] V. M. Miklyukov and V. D. Vedenyapin, Extrinsic dimension of tubular minimal hypersurfaces, Mat. Sb. 131 (1986), 240–250; English transl. in Math. USSR Sb. 59 (1988).

[13] J. C. C. Nitsche, A uniqueness theorem of Bernstein's type for minimal surfaces in cylindrical coordinates, J. Math. Mech. 11 (3) (1962), 293–302.

[14] R. Osserman, A survey of minimal surfaces, Dover Publications, New York 1987.

[15] N. O. Pyleva and V. G. Tkachev, Projective volume of minimal graphs, in preparation.

[16] L. Simon, A Hölder estimate for quasiconformal mappings between surfaces in Euclidean space, with application to graphs, having quasiconformal Gauss map, Acta Math. 139 (1977), 19–51.

[17] L. Simon, Equation of mean curvature type in two independent variables, Pacif. J. Math. 69 (1977), 245–268.

[18] V. G. Tkachev, The external estimates of clasp function for elliptic hypersurfaces, Preprint 2031-B92, deposited at VINITI, 1992, 17 pp. (in Russian).

[19] V. G. Tkachev, Minimal tubes and coefficients of holomorphic functions in an annulus, Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform. XLV (1995), 19–26.