A SHARP LOWER BOUND FOR THE FIRST EIGENVALUE ON A MINIMAL SURFACE

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It is well known that minimal surfaces in Euclidean spaces inherit many properties of planes. Under certain conditions on the surface topology or the geometric structure of its boundary, one can assert the validity of the classical isoperimetric inequality (see [1, 2]). In [3] E.Giusti proved "the mean value theorem" for subharmonic functions on minimal graphs. Other examples concerning geometric inequalities and their applications to surfaces of prescribed mean curvature can be found in [4, 5].

The aim of the present note is to study the properties of the first eigenvalue of a given open subset on a minimal submanifold in $\mathbb{R}^N$. Following [6], we call this characteristic *the principal frequency* of the set.

Let $\mathcal{M}$ be a surface in $\mathbb{R}^N$, given by a C^2-smooth isometric immersion $x(m)$ of a p-dimensional manifold M. We shall denote by ∇, Δ, and H the covariant derivative, the Laplace–Beltrami operator on $\mathcal{M}$, and the mean curvature vector of the immersion x, respectively. For a given open subset $\mathcal{D} \subset M$ and a point $a \in \mathbb{R}^N$ we introduce the following characteristic of growth of the mean curvature of the surface $\mathcal{M}$, by setting

$$k_a(\mathcal{D}) = \inf_{m\in\mathcal{D}} \langle x(m) - a, H(m)\rangle,$$

where the angle brackets denote the standard scalar product in $\mathbb{R}^N$. If, in addition, the set $\mathcal{D}$ has a non-empty piecewise smooth boundary $\partial\mathcal{D}$, then its principal frequency $\lambda\mathcal{D}$ is defined as

$$\lambda(\mathcal{D}) = \inf \frac{\int_{\mathcal{D}} |\nabla\varphi|^2\, dM}{\int_{\mathcal{D}} \varphi^2\, dM}, \tag{1}$$

where the infimum is taken over all Lipschitz continuous functions φ vanishing on $\partial\mathcal{D}$. As a simple corollary of the definition, we note that the principal frequency is a non-increasing set function. That is, if $\mathcal{D}_1 \subset \mathcal{D}_2$ are subsets of M with the above-mentioned properties, then $\lambda(\mathcal{D}_1) \geq \lambda(\mathcal{D}_2)$. This property allows one to extend the class of λ-measurable subsets in M by defining the principal frequency of an arbitrary open subset $\mathcal{D}$ with non-empty boundary

$$\lambda(\mathcal{D}) = \inf\{\lambda(\mathcal{D}_1) : \mathcal{D}_1 \subset \mathcal{D},\ \partial\mathcal{D} \in C^1\}.$$

It is well known that if $\partial\mathcal{D}$ is sufficiently smooth, then $\lambda(\mathcal{D})$ is the smallest positive λ satisfying

$$\Delta f = -\lambda f$$

for some function $f(m)$ which is C^2-smooth in $\mathcal{D}$ and vanishes on $\partial\mathcal{D}$.

We denote by $B_a(R)$ a ball in $\mathbb{R}^N$ of radius R centered at a. Then, for an arbitrary $\mathcal{D} \subset M$, there is only one way to define the smallest ball (with respect to inclusion) $B_a(R) \supset x(M)$, which we say is *circumscribed* around $\mathcal{D}$ (here $R = \infty$ is also possible). By $\rho(\mathcal{D})$ we denote the radius of this ball. Then, since the immersion is isometric, we can conclude that

$$\rho(\mathcal{D}) \leq d(\mathcal{D}), \tag{2}$$

where $d(\mathcal{D})$ denotes the radius of the smallest geodesic ball containing $\mathcal{D}$. To make the statements less cumbersome, we shall say that $\mathcal{D}$ is an open subset of the surface $\mathcal{M}$ if there exists an open $\mathcal{D}_1 \subset M$ such that $\mathcal{D} = x(\mathcal{D}_1)$.

We now formulate the main result of the paper.

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Theorem 1. *Let $\mathcal{M}$ be a p-dimensional surface in $\mathbb{R}^N$, and let $\mathcal{D} \subset \mathcal{M}$ be an arbitrary open set such that $\rho(\mathcal{D}) < \infty$ and $k_a(\mathcal{D}) \geq 1 - p$, where a is the center of the ball circumscribed to $\mathcal{D}$. Then*

$$\lambda(\mathcal{D}) \geq \frac{\mu(\nu)}{\rho^2(\mathcal{D})}, \tag{3}$$

where $\nu = p + [k_a(\mathcal{D})]$ and $\mu(\nu)$ stands for the principal frequency of the unit ν-dimensional Euclidean ball.

Remark. We note that, due to relation (2), our inequality is stronger than the already known results (cf. [4, 5]) based on the use of geodesic distance.

We recall that the surface $\mathcal{M}$ is called *minimal* if its mean curvature vector vanishes identically. As an important application of the above theorem we get then

Corollary 1. *Let $\mathcal{M}$ be a p-dimensional submanifold in $\mathbb{R}^N$. Then for any open subset $\mathcal{D} \subset \mathcal{M}$,*

$$\lambda(\mathcal{D}) \geq \frac{\mu(p)}{\rho^2(\mathcal{D})}. \tag{4}$$

Moreover, the equality is attained only in the case of a p-dimensional plane in $\mathbb{R}^N$ passing through the center of the ball circumscribed around $\mathcal{D}$.

Well known is the classical problem of discovering conditions for realization of a given metric on a surface which is immersed into $\mathbb{R}^N$ and has a number of prescribed properties. The following assertion gives an obstruction of a sort to solution of this problem in terms of the growth rate of the principal frequencies of geodesic balls. It is worth noting that here the role of the obstruction is played by internal invariants of the metric.

Corollary 2. *Let $\mathcal{D}$ be a domain in $\mathbb{R}^p$ endowed with a metric ds^2 that is complete in the domain. Let $z_0 \in \mathcal{D}$ be a fixed point, and let $\mathcal{D}(R)$ be a geodesic ball of radius R centered at z_0. Then, if*

$$\lim_{R \to +\infty} \lambda(\mathcal{D}(R))R^2 < \mu(p),$$

the pair $(\mathcal{D}, ds^2)$ cannot be realized as a minimal submanifold in $\mathbb{R}^N$ for $N \geq p$.

The proof of corollary 2 follows directly from (2) and (4).

E.Calabi in [7] made a conjecture according to which no internally complete minimal surface in $\mathbb{R}^3$, other than the plane, can be contained in a half-space or a ball. In [8] Xavier and Jorge constructed a counterexample to this conjecture in the case of a half-space, and in [9] they showed that if a complete minimal surface has a bounded Gauss curvature then it cannot be a bounded subset of $\mathbb{R}^N$.

Corollary 3. *Let $\mathcal{M}$ be an internally complete surface with $H = 0$ in $\mathbb{R}^N$ which is contained in some Euclidean ball of radius R. Then the first eigenvalue of the manifold M is non-zero and*

$$\lambda_1(M) \geq \frac{\mu(p)}{R^2}.$$

The results of Cheng and Yau [10] imply in this case, in particular, that the growth rate of the areas of the geodesic balls on such a surface must be must be higher than polynomial.

Remark 2. We note that the bound (4) presupposes no topological or geometric assumptions on the surface in addition to minimality of the immersion. Due to the above-mentioned monotonicity property of the first eigenvalue, (4) shows that the part of a minimal surface contained in a given Euclidean ball cannot be "too abundant." It seems interesting to compare this observation with the well-known "monotonicity formula" for minimal surfaces (see, e.g., [5, p. 63]) that gives an a priori lower bound for the area of such a surface passing through the origin.

The proof of Theorem 1 is based on a result of Cheng and Yau [10] which we formulate in Section 2 and on some special properties of the solutions of the Bessel equation.

1. Auxiliary assertions. The main assertion of this section is

Theorem 2. *Let $\nu \geq 1$ be an arbitrary integer, $\lambda > 0$, and let $Y(x)$ be a solution of the equation*

$$xY''(x) + (\nu - 1)Y'(x) + \lambda x Y(x) = 0 \tag{5}$$

on the segment $[0, R]$, with boundary conditions

$$Y(0) = 1, \qquad Y'(0) = 0, \tag{6}$$

such that

$$Y(x) > 0 \qquad \forall x \in [0, R). \tag{7}$$

Then the following inequality holds everywhere on $(0, R)$:

$$xY''(x) - Y'(x) > 0. \tag{8}$$

First we prove the following statements concerning the properties of the solutions of the problem (5)–(7).

Lemma 1. *If $Y(x)$ is a solution of (5)–(7), then the function*

$$Y_1(x) = -\frac{1}{x}Y'(x)$$

also is a solution of (5) for $\nu_1 = \nu + 2$ and $\lambda_1 = \lambda$. Moreover,

$$\lim_{x \to +0} Y_1(x) = \frac{\lambda}{\nu} > 0.$$

Proof. We note that (5) implies

$$0 = Y''(0) + (\nu - 1) \lim_{x \to +0} \frac{1}{x} Y'(x) + \lambda Y(0) = \lambda + \nu Y''(0),$$

i.e., $Y''(0) = -\lambda/\nu$. Therefrom one can find

$$\lim_{x \to +0} Y_1(x) = \lim_{x \to +0} \left(-\frac{1}{x}Y'(x)\right) = Y''(0) = \frac{\lambda}{\nu}.$$

We now check that $Y_1(x)$ is a solution of (5) on the interval $(0, R)$. In fact, we can see directly that

$$Y_1' = \frac{1}{x^2}(Y' - Y''x), \qquad \lim_{x \to +0} Y_1'(x) = 0,$$
$$Y_1'' = \frac{1}{x^3}(2xY'' - 2Y' - x^2 Y'''),$$

and so

$$xY_1'' + (\nu + 1)Y_1' + \lambda x Y_1 = -Y''' - \frac{\nu - 1}{x}Y'' + Y'\left(\frac{\lambda x^2 + \nu - 1}{x^2}\right)$$
$$= \frac{d}{dx}\left(Y'' + \frac{\nu - 1}{x}Y' + \lambda Y\right).$$

Now the necessary relation follows from (5), and Lemma 1 is proved.

In accordance with the above lemma, one can consider the new solution Y_1 to be extended to the whole of $[0, R]$.

Lemma 2. *Every solution of* (5)–(7) *is a strictly monotone function on* $[0, R]$.

Proof. In the case $\nu = 1$ we have $Y(x) = \cos\dfrac{\pi x}{2c}$ for some $c \geq R$, and the assertion of Lemma 2 is evident.

Let $\nu \geq 2$. We first show that $Y(x)$ satisfies the minimum principle on the segment $[0, R)$. If this were not so, there would exist $\xi \in [0, R)$ which is a point of local minimum of $Y(x)$. This implies that

$$Y'(\xi) = 0 \quad \text{and} \quad Y''(\xi) \geq 0.$$

From the proof of the previous lemma it follows that $Y''(0) = -\frac{\lambda}{\nu} < 0$, i.e., $\xi > 0$. Taking (5) into account, we get

$$Y''(\xi) + \lambda Y(\xi) = 0,$$

which, together with the non-negativity of $Y''(\xi)$, gives $Y(\xi) \leq 0$. This contradicts assumption (7).

Using once more the non-negativity of the second derivative of the solution at the origin and the boundary condition (6), we get that $Y(x)$ is strictly decreasing in some neighborhood of the origin and so, due to the minimum principle, it is at any rate a non-increasing function. We now prove that $Y'(x) \neq 0$ on $(0, R)$, or equivalently $Y'(x) < 0$ everywhere on $(0, R)$.

To this end we introduce an auxiliary function $Y_1(x) = -Y'(x)/x$, which by Lemma 1 is a solution of (5) for $\nu_1 = \nu + 2$. Due to what we have established above, $Y'(x) \leq 0$ for $x \in [0, R]$ and so $Y_1(x) \geq 0$ and $Y_1(0) = \frac{\lambda}{\nu}$.

Suppose that $Y_1(\xi) = 0$ at some point $\xi \in [0, R)$. Then $\xi > 0$ and by virtue of the theorem of existence and uniqueness for the solutions of the second-order equation

$$Y_1''(x) = -\frac{\nu + 1}{x} Y_1'(x) - \lambda Y_1(x), \qquad \forall x \in (0, R],$$

either $Y_1'(\xi) \neq 0$, or $Y_1(x)$ is identically zero. Since the latter is impossible, we conclude that the function $Y_1(x)$ changes sign in the neighborhood of ξ. This contradicts the non-negativity of $Y_1(x)$, hence $Y_1(x) > 0$ everywhere on $[0, R)$ i.e., $Y(x)$ is strictly decreasing.

Proof of Theorem 2. We set $v(x) = Y''x - Y'$ or

$$v(x) = x^3 D^2[Y],$$

where D is a first order differential operator of the form

$$D[f] = -\frac{1}{x}\frac{d}{dx}(f(x)).$$

By virtue of Lemma 1, the function $Y_2 = D^2[Y]$ is a solution of (5) for $\nu_2 = \nu + 4$. Moreover, because of Lemmas 1 and 2, the functions $Y_n(x) = D^n[Y]$ are strictly positive for $n \geq 1$, which can be easily deduced by induction from (6) and (7). Thus

$$v(x) = x^3 Y_2(x) > 0$$

everywhere on the half-interval $(0, R]$, q.e.d.

2. Proof of Theorem 1. It will be convenient to carry out the proof of Theorem 1 in a more general situation, suitable for other applications. Let $V \subset \mathbb{R}^N$ be a fixed n-dimensional subspace, let $\{v_i\}_{i=1}^n$ be some orthonormal basis of V, and let $f^2(m) = \sum_{i=1}^n \langle x(m), v_i \rangle^2$. It is clear that the value of $f(m)$ is independent of the choice of the basis.

Lemma 3. *The following relations hold:*

$$\nabla f(m) = \frac{1}{f(m)} v^{\top} \tag{9}$$

and

$$\Delta f(m) = \frac{\sigma(V,T) - 1}{f(m)} + \frac{\langle H(m), v\rangle}{f(m)} + \frac{|v^{\perp}|^2}{f^3(m)}, \tag{10}$$

where $v = \sum_{i=1}^{n} v_i \langle v_i, x(m)\rangle$, $\sigma(V,T) = \sum_{i-1}^{n} |v_i^{\top}|^2$ *and the symbols* $\perp$ *and* $\top$ *denote the projections of a given vector on the tangent and the normal space of* $\mathcal{M}$*, respectively.*

Proof. Let E be a tangent vector to $\mathcal{M}$, and let $\overline{\nabla}$ be a canonical covariant derivative in $\mathbb{R}^N$. Then

$$\begin{aligned}\nabla_E f(m) &= \frac{1}{f(m)} \sum_{i=1}^{n} \langle v_i, x(m)\rangle \nabla_E \langle x(m), v_i\rangle \\ &= \frac{1}{f(m)} \sum_{i=1}^{n} \langle v_i, \overline{\nabla}_E x(m)\rangle \langle x(m), v_i\rangle \\ &= \frac{1}{f(m)} \sum_{i=1}^{n} \langle v_i^{\top}, E\rangle \langle v_i, x(m)\rangle = \frac{\langle E, v^{\top}\rangle}{f(m)}\end{aligned}$$

and by definition of the gradient we get (9).

We note further that for any $e \in \mathbb{R}^N$ we have the following relation between the Laplace-Beltrami operator and the mean curvature vector (see [11, p. 309]):

$$\Delta\langle x(m), e\rangle = \langle H(m), e\rangle,$$

and so

$$\begin{aligned}\frac{1}{2}\Delta f^2(m) &= \sum_{i=1}^{n} \left[\langle x(m), v_i\rangle \Delta\langle x(m), v_i\rangle + |\nabla\langle x(m), v_i\rangle|^2\right] \\ &= \sum_{i=1}^{n} \langle x(m), v_i\rangle\langle H(m), v_i\rangle + \sum_{i=1}^{n} |e_i^{\top}|^2 \\ &= \langle v(m), H(m)\rangle + \sigma(V,T).\end{aligned} \tag{11}$$

On the other hand,

$$\frac{1}{2}\Delta f^2(m) = f(m)\Delta f(m) + |\nabla f|^2 = f(m)\Delta f(m) + \frac{|v^{\top}|^2}{f^2(m)}$$

and, by virtue of (11), we establish (10).

Lemma 4. *Let* $\mathcal{M}$ *be a* p*-dimensional surface in* $\mathbb{R}^N$ *with mean curvature vector* H*. Let* V *be an* n*-dimensional subspace of* $\mathbb{R}^N$*, and let* $C_a^V(R)$ *be a generalized cylinder of the form*

$$C_a^V(R) = \{x \in \mathbb{R}^N : |x^V - a| < R\},$$

where $a \in V$ *and* x^V *is the projection of the vector* x *on* V*. Let* $k = k_a(\mathcal{M} \cap C_a^V(R))$*, and suppose that*

$$\nu = n + p + [k] - N \geq 1 \tag{12}$$

holds. Then

$$\lambda(\mathcal{M} \cap C_a^V(R)) \geq \frac{\mu(\nu)}{R^2}. \tag{13}$$

Proof. We shall assume, without loss of generality, that $a = 0$. We note first that

$$\sigma(V,T) = \sum_{i=1}^{n} |v_i^{\top}|^2 = n - \sum_{i=1}^{n} |v_i^{\perp}|^2 \geq n - (N-p),$$

whence, taking into account the definition of k and using (10), we get

$$\Delta f \geq \frac{n+p+k-1}{f(m)} + \frac{|v^{\perp}|^2}{f^3(m)} \geq \frac{\nu - 1}{f(m)} + \frac{|v^{\perp}|^2}{f^3(m)}. \tag{14}$$

We now fix a solution $Y(t)$ of the problem (5)–(7) which corresponds to the parameter value $\nu = n + p + [k] - N$, with λ_0 chosen so that the condition $Y(R) = 0$ is fulfilled. Namely, due to homogeneity of (5), it is easy to check that $\lambda_0 = \mu(p)/R^2$. Using the negativity of the derivative $Y'(t)$ and relation (14) we get for the composition $\varphi = Y_0 f(m)$

$$\begin{aligned} \Delta\varphi(m) &= Y'(f)\Delta f + |\nabla f|^2 Y''(t) \\ &\leq \frac{Y'(f)}{f(m)}(\nu - 1) + \left(\frac{Y'(f)}{f(m)} - Y''(f(m))\right)\frac{|v^{\perp}|^2}{f^2(m)} + Y''(f) \\ &= -\lambda_0 Y(f) + \frac{|v^{\perp}|^2}{f^3(m)}(Y'(f) - fY''(f)). \end{aligned} \tag{15}$$

In [10] Cheng and Yau pointed out that *if $\varphi(m)$ is twice differentiable and $\mathcal{D} \subset M$, then*

$$\lambda(\mathcal{D}) \geq \inf_{m \in \mathcal{D}} \left(-\frac{\Delta\varphi(m)}{\varphi(m)}\right). \tag{16}$$

Taking the set $x^{-1}(\mathcal{M} \cap C_a^V(R))$ as $\mathcal{D}$ and combining (15) and (16), we have

$$\lambda(\mathcal{M} \cap C_a^V(R)) \geq \lambda_0 + \frac{|v^{\perp}|^2}{f^3(m)}(Y'(f) - fY''(f)). \tag{17}$$

Inequality (13) now follows from Theorem 2 and the definition of λ_0.

The proof of Theorem 1 immediately follows from the above lemma if we put $V = \mathbb{R}^N$, and so $C_a^V(R) = B_a(R)$.

We note that (4) becomes an equality only if we have $|x^{\perp}(m)| = 0$ everywhere on $\mathcal{M} \cap B_a(R)$. However, this means that the radius vector $x(m)$ of the immersion is tangent to $\mathcal{M}$ at every point, i.e., that the surface $\mathcal{M}$ is a cone centered at a. Taking into account the regularity of the immersion, we conclude that such a surface coincides with a p-dimensional plane passing through a.

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