On the Jacobian of the harmonic moment map

Björn Gustafsson and Vladimir Tkachev

Abstract. In this paper we represent harmonic moments in the language of transfinite functions, that is projective limits of polynomials in infinitely many variables. We obtain also an explicit formula for the Jacobian of a generalized harmonic moment map.

Introduction

With any integer k and any closed oriented analytic curve Γ one can associate the kth harmonic moment of Γ, defined as

$$M_k(\Gamma) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma z^k \bar{z}\, dz.$$

When k is nonnegative and $\Gamma = \partial\Omega$ for some domain Ω, the moment takes the more familiar form

$$M_k(\Gamma) = \frac{1}{\pi} \iint_\Omega z^k dxdy, \qquad z = x + \mathrm{i}y.$$

Information about Γ can be read off from these harmonic moments, which have turned out to be useful geometric objects in many problems of complex analysis and potential theory.

The harmonic moments appear also in algebraic contexts, in the first place in connection with conservation laws discovered in 40's by P. Polubarinova-Kochina [15] and L. Galin [3], and were also studied in 70's by S. Richardson [17] in application to the Hele-Shaw problem (see [9] for a full account of relevant material). It turns out that the quantities $M_k(\Gamma)$ constitute a hierarchy of conservative quantities for the boundary Γ under the action of Hele-Shaw flow with a source at the origin (Laplacian growth). This allows to describe Hele-Shaw evolution explicitly for a wide class of polynomial domains. This complete integrability has recently

The authors have been supported by supported by the Swedish Research Council, grant KAW 2005.0098 from the Knut and Alice Wallenberg Foundation, and by the European Science Foundation Networking Programme HCAA.

been subject to new investigations by I. Krichever, A. Marshakov, M. Mineev-Weinstein, P. Wiegmann and A. Zabrodin, e.g. [21], [10], [11]. One point of view is that harmonic moments can be thought of as canonical coordinates in certain well established integrable hierarchies (for example, in the dispersionless 2D Toda hierarchy), and it has for example been shown that they can be written as derivatives of an associated tau-function.

Despite the above mentioned applications, some principal difficulties remain on the level of mathematical rigor with the basic definitions of harmonic moments regarded as functionals in infinitely many variables. In particular, there is no established algebraic or analytic calculus which allows importing harmonic moments as well-defined functionals. In this paper we make an attempt to represent harmonic moments in the language of transfinite functions, that is projective limits of polynomials in infinitely many variables. In this picture, not only the usual complex variables (z etc.), but also the coefficients $(a_0, a_1, \dots)$ of analytic functions are treated as variables. Some analogies with symmetric functions or germs of analytic functions may be traced. For example, the number of variables is irrelevant in symmetric functions and any symmetric function is uniquely determined if the number of variables is large enough. A similar rigidity is valid for harmonic moments in the sense that they stabilize after truncation of variables of higher grade.

We mention that although the model of transfinite functions discussed below allows manipulation of objects in an algebraic manner, it does not allow a priori to 'evaluate' the objects. In some particular cases, for example for domains which are conformal images of univalent polynomials, the transfinite calculus becomes finite, which makes direct evaluations possible. In more involved cases one needs an adequate homomorphism into one of standard evaluation rings. However, we will not pursue this matter in the present article.

Another, more concrete, application of the above formalism is an explicit formula for the Jacobian of a generalized harmonic moment map.

1. Transfinite functions

1.1. Projective limits of polynomial rings

Let R be a commutative ring with unit and A a set of independent commutating[1] variables. Then the polynomial ring $R[A]$ makes sense, even if the set A is infinite (of any cardinality). We shall introduce transfinite functions over A as formal sums (in general infinite) of monomials in the variables A with coefficients in R such that for each finite subset $F \subset A$ there are only finitely many terms which contain variables only from F. In other words, by setting every variable in $A \setminus F$ equal to zero a transfinite function reduces to a polynomial in $R[F]$.

[1] In the paper we deal only with commutating variables while all the constructions below are still valid for non-commutative setup without any changes.

For the formal definition the notion of projective limit is appropriate. Let $\mathcal{F}$ denote the family of finite subsets of A. This is a directed set in a natural way: it is partially ordered by inclusion ($F_1 \subset F_2$, $F_1, F_2 \in \mathcal{F}$) and with this partial order any two elements have an upper bound, namely their union (if $F_1, F_2 \in \mathcal{F}$ then $F = F_1 \cup F_2 \in \mathcal{F}$ and $F_1 \subset F$, $F_2 \subset F$). These inclusions induce projection homomorphisms

$$\pi_{F_1,F_2} : R[A]/(A \setminus F_2) \to R[A]/(A \setminus F_1), \tag{1.1}$$

where, for any subset $S \subset R[A]$, (S) denotes the ideal in $R[A]$ generated by S.

The maps (1.1) define a projective system of rings based on the directed set $\mathcal{F}$. We define transfinite functions by passing to the projective (or inverse) limit.

Definition 1.1. The ring of *transfinite* functions over A and with coefficients in R is the projective limit

$$R_\infty[A] = \lim_{\leftarrow F} R[A]/(A \setminus F).$$

Recall that, as a set, the projective limit $R_\infty[A]$ can be taken to be that subset of the cartesian product $\Pi_{F\in\mathcal{F}} R[A]/(A \setminus F)$ for which the F_1's and F_2's components are related by the map π_{F_2,F_1} whenever $F_1 \subset F_2$. It is actually enough to consider only cofinal segments in the above cartesian product, because for any $F \in \mathcal{F}$, the F's component of an element determines uniquely the F_1's, for any $F_1 \subset F$. In some operations with the projective limit the so arising possibility of self-correction of initial segments is important. One may think of $R_\infty[A]$ as a kind of completion of $R[A]$.

The above definition of transfinite functions is modeled on standard definitions of p-adic numbers and formal power series. For example, the ring of formal power series is $\mathbb{C}[[z]] = \lim_{\leftarrow n} \mathbb{C}[z]/(z^n)$. However, the transfinite functions are actually somewhat simpler, because for any $F \in \mathcal{F}$ there is a natural embedding $R[F] \to R[A]$ which becomes an isomorphism

$$R[F] \cong R[A]/(A \setminus F).$$

The projection maps (1.1) therefore have the alternative description as maps

$$\pi_{F_1,F_2} : R[F_2] \to R[F_1], \tag{1.2}$$

most easily described by declaring that all variables in $F_2 \setminus F_1$ shall be set equal to zero.

The natural projection maps

$$\pi_F : R_\infty[A] \to R[A]/(A \setminus F).$$

correspond in this simpler picture to the previously mentioned maps $\pi_F : R_\infty[A] \to R[F]$ (setting all variables not in F equal to zero). In the other direction we have inclusion maps $R[F] \to R_\infty[A]$, which postcomposed with the π_F give the identity maps on the $R[F]$.

The definition with projection maps (1.1) is somewhat more flexible than (1.2) in that it allows for including formal power series (with coefficients in R) into the picture, namely by admitting powers a^n ($a \in A$) among the generators of the ideals. However, we shall not go into such generalizations in the present paper, and therefore we shall work in the simpler setting (1.2).

1.2. Transfinite functions

The typical way we shall use the above general definitions is as follows. We are interested in forming transfinite functions in the variables z, $a_0, a_1, a_2 \dots$, plus sometimes corresponding conjugated variables ($\bar{a}_1$ etc.) and variables with negative index (a_{-1} etc.), all treated as independent variables. We also want to be able to invert a few of the variables, namely z, a_0, i.e. admit z^{-1} and a_0^{-1}. Hence these cannot be set equal to zero. We can achieve the above goals by choosing the ground ring R to contain the variables we want to invert, namely we take

$$R = \mathbb{C}[z, z^{-1}, a_0, a_0^{-1}] := \mathbb{C}[z, w, a_0, b_0]/(zw - 1, a_0 b_0 - 1).$$

Then our ring of transfinite functions will be

$$\mathbb{C}_\infty[z, z^{-1}, a_0, a_0^{-1}; a_1, a_2, \dots] := R_\infty[a_1, a_2, \dots].$$

Set

$$A_0 = \{z, z^{-1}, a_0, a_0^{-1}\}$$

for the "*null-variables*", or invertible variables, and

$$A_n = \{a_1, a_2, \dots, a_n\}$$

for the remaining variables up to some index $n \geq 1$. This set could, depending on the context, also contain the corresponding conjugated variables (considered as independent variables), or variables with negative index, for example $A_n = \{a_1, \bar{a}_1, a_2, \bar{a}_2, \dots, a_n, \bar{a}_n, \}$ or $A_n = \{a_{-n}, \dots, a_{-2}, a_{-1}, a_1, a_2, \dots, a_n\}$. Finally, set

$$A_+ = \bigcup_{n\geq 1} A_n, \quad A = A_0 \cup A_+.$$

The ring of transfinite functions will in this context be denoted

$$\mathbb{C}_\infty[A] = \mathbb{C}_\infty[z, z^{-1}, a_0, a_0^{-1}; a_1, a_2, a_3, \dots] = R_\infty[A_+].$$

In the construction of $\mathbb{C}_\infty[A]$ as a projective limit it is enough to use the sets A_n (in place of all finite subsets of A). Thus, setting

$$\mathbb{C}_n[A] = \mathbb{C}[z, z^{-1}, a_0, a_0^{-1}; a_1, a_2, \dots, a_n] = R[A_n],$$

we have the projective system $\{\mathbb{C}_n[A]\}_{n\geq 0}$, with projection homomorphisms

$$\pi_{ij} : \mathbb{C}_j[A] \to \mathbb{C}_i[A], \qquad j \geq i \geq 0$$

defined by setting the variables in $A_j \setminus A_i$ equal to zero (equivalently, by removing every term which contains such a variable). The projective limit is

$$\mathbb{C}_\infty[A] = \lim_{\leftarrow n} \mathbb{C}_n[A]$$

with projections

$$\pi_n : \mathbb{C}_\infty[A] \to \mathbb{C}_n[A],$$

obtained by setting the variables in $A_+ \setminus A_n$ equal to zero.

In case conjugate variables are included there is a natural involution

$$^* : \mathbb{C}_\infty[A] \to \mathbb{C}_\infty[A]$$

which for any variable which has a conjugate exchanges the two (e.g., $a_1 \mapsto \bar{a}_1$, $\bar{a}_1 \mapsto a_1$). Variables which do not have a conjugate should be thought of as real variables, and remain unchanged under the involution.

One way to describe the ring of transfinite functions is to consider all sequences $(h_n)_{n\geq 0}$ of elements in $\mathbb{C}_n[A]$ stable under high-order substitutions

$$\pi_{ij}(h_j) = h_i, \qquad j \geq i \geq 0, \tag{1.3}$$

and equipped with pointwise algebraic operations. In what follows we make no distinction between such sequences and their limits in $\mathbb{C}_\infty[A]$ and write $h = \underset{\leftarrow}{\lim}\, h_n$. The element h_n is called the nth approximant of h. The described (projective) convergence is rigid in the sense that any approximant h_n is determined uniquely by h.

In the other direction we have the injection $\mathbb{C}_n[A] \to \mathbb{C}_\infty[A]$, by which any element $f \in \mathbb{C}_n[A]$ gives rise to a transfinite function by taking the inverse limit, $\underset{\leftarrow}{\lim}\, f_k$, of the sequence

$$f_k := \begin{cases} \pi_{kn}(f), & \text{for } 0 \leq k \leq n; \\ f, & \text{for } k \geq n \end{cases}$$

This $\underset{\leftarrow}{\lim}\, f_k$ is the unique element in $\mathbb{C}_\infty[A]$ with the property that its nth approximant is exactly f. In what follows we identify 'finite' polynomials from $\mathbb{C}_n[A]$ and their lifting in $\mathbb{C}_\infty[A]$ (that is consider the $\mathbb{C}_n[A]$ as subrings in $\mathbb{C}_\infty[A]$).

It is natural consider a transfinite function h as a function in an infinite number of variables. Although we think of all the variables A as 'complex' variables and 'variable coefficients', no value ('number') can be assigned to h at any concrete point. However, one is allowed to substitute finitely many variables by complex numbers (this is a special case of operation 2) in the next subsection), and the result will be another transfinite function.

In order to pass completely from transfinite objects to classical ones, however, one need to have an evaluation on $\mathbb{C}_\infty[A]$, like the limit in topological categories. Algebraically this is equivalent to constructing an adequate homomorphism from the ring $\mathbb{C}_\infty[A]$ to some standard evaluation ring. A choice of such a homomorphism should be made individually in each concrete case.

Example 1. The simplest example of a transfinite function is the *transfinite power series* in z, with variable coefficients a_k:

$$\underset{\leftarrow}{\lim} \sum_{k=0}^{n} a_k z^{k+1} = \sum_{k=0}^{\infty} a_k z^{k+1}. \tag{1.4}$$

This is an element in $\mathbb{C}_\infty[z, a_0, a_1, \dots]$. On the other hand, a power series with constant coefficients, like $\sum_{k=0}^{\infty} k! z^k$, is not a transfinite function in our sense, but is an element of $\mathbb{C}[[z]] = \lim_{\leftarrow n} \mathbb{C}[z]/(z^n)$.

1.3. Operations with transfinite functions

1) (*General maps.*) Consider a map $q : \mathbb{C}[A] \to \mathbb{C}[A]$ which commutes with restrictions for large enough indices:

$$q \circ \pi_{ij} = \pi_{ij} \circ q, \qquad i \geq m,$$

where $m \geq 0$ depends only on q. Then q extends naturally to a map on $\mathbb{C}_\infty[A]$ as follows. Let $x \in \mathbb{C}_\infty[A]$ and $x_n \in \mathbb{C}_n[A]$ be the sequence of its approximants. Define $y_i = q(x_i)$ for $i \geq m$ and $y_i = \pi_{ij}(y_m)$ for $i \leq m$. Then $(y_n)_{n\geq 0}$ satisfies (1.3) because for any $j \geq i \geq m$

$$\pi_{ij}(y_j) = \pi_{ij}(q(x_j)) = q(\pi_{ij}(x_j)) = q(x_i) = y_i$$

and similarly one checks (1.3) for small indices. Therefore $(y_n)_{n\geq 0}$ induces en element in $\mathbb{C}_\infty[A]$ denoted by $q(x)$.

2) (*Substitution.*) Let A, B be two families of variables, filtered by A_n, B_n $(n \geq 0)$ respectively, let $\phi \in \mathbb{C}_\infty[B]$, let $X \subset B$ be a finite subset and let $t : X \to \mathbb{C}_\infty[A]$ be any map. Then for any $i \geq 0$ we define

$$\psi_i = \phi_i|_{(t,X)} \in \mathbb{C}_i[A, B \setminus X],$$

where $|_{(t,X)}$ means that one makes the substitutions $x = (t(x))_n = \pi_n \circ t(x) \in \mathbb{C}_n[A]$ for each $x \in X$. This new sequence obviously satisfies (1.3), hence induces an element in $\mathbb{C}_\infty[A, B \setminus X]$, which we denote by $\phi|_{(t,X)}$.

Remark 1.2. The introduced composition of two transfinite functions is very close to what is known as the plethysm in category of symmetric functions (see, e.g., [13, p. 135].

3) (*Derivation.*) Another example is the partial derivative. For any variable $x \in A_m$

$$\partial_x \circ \pi_{ij} = \pi_{ij} \circ \partial_x, \qquad i \geq m.$$

Therefore ∂_x extends to $\partial_x : \mathbb{C}_\infty[A] \to \mathbb{C}_\infty[A]$. One then checks easily that ∂_x is a derivation on $\mathbb{C}_\infty[A]$, that is ∂_x is linear and satisfies the Leibniz rule. Moreover, the derivatives satisfy the usual commutativity: for any x, y

$$\partial_x \partial_y = \partial_y \partial_x.$$

4) (*Coefficient extraction.*) An important operation is coefficient extraction. Let $y \in \mathbb{C}_\infty[A]$ and $x \in A$. Then for any fixed n the coefficient $[x^n](y_i)$ of x^n in $y_i \in \mathbb{C}_i[A]$ is well defined and extends in an obvious way to an element $[x^n](y) \in \mathbb{C}_\infty[A]$.

1.4. The transfinite resultant

Recall that the resultant of two polynomials $f(z) = a_n \prod_{i=1}^n (z - \xi_i) = \sum_{i=0}^n a_i z^i$ and $g(z) = b_m \prod_{j=1}^m (z - \eta_j) = \sum_{j=0}^m b_j z^j$ is a polynomial function in the coefficients of f and g having the elimination property that it vanishes if and only if f and g

have a common zero [20], [4]. In terms of the zeros of the polynomials the resultant is given by the Poisson product formula

$$\mathcal{R}_{\mathrm{pol}}(A, B) = a_n^m b_m^n \prod_{i,j}(\xi_i - \eta_j) = a_n^m \prod_{i=1}^{n} g(\xi_i).$$

Alternatively, the resultant can be computed as the determinant of the Sylvester matrix of size $n + m$

$$\mathcal{R}_{\mathrm{pol}}(f, g) = \det \begin{pmatrix} a_0 & a_1 & \dots & a_n & & \\ & \dots & \dots & \dots & \dots & \\ & & a_0 & a_1 & \dots & a_n \\ b_0 & b_1 & \dots & b_m & & \\ & \dots & \dots & \dots & \dots & \\ & & b_0 & b_1 & \dots & b_m \end{pmatrix},$$

where the first m rows are the shifted coefficients of f, the next n rows are the shifted coefficients of g.

The authors introduced recently [8] a notion of the meromorphic resultant of two meromorphic functions on an arbitrary compact Riemann surface. For any two meromorphic functions f and g, whose divisors and have no common points, the number

$$\mathcal{R}(f, g) = g((f)) \equiv \prod_i g(\xi_i)^{N_i},$$

is called the meromorphic resultant of f and g. Here $(f) = \sum_i N_i \cdot \xi_i$ is the divisor of f. For the general properties of the meromorphic resultant, see [8]. We mention only that the meromorphic resultant is symmetric and homogeneous of degree zero (i.e. depends only on the divisors of f and g). Moreover, for two rational functions $f(z) = \sum_{k=0}^{n} b_k z^{-k}$ and $g(z) = \sum_{k=0}^{m} a_k z^k$, one easily finds that their meromorphic resultant is related to the classical polynomial resultant by the formula:

$$\begin{aligned} \mathcal{R}(\sum_{k=0}^{n} b_k z^{-k}, \sum_{k=0}^{m} a_k z^k) &= \frac{1}{a_0^n b_0^m} \mathcal{R}_{\mathrm{pol}}(\sum_{k=0}^{n} b_k z^{n-k}, \sum_{k=0}^{m} a_k z^k) \\ &= \frac{1}{a_0^n b_0^m} \det \begin{pmatrix} b_0 & b_1 & \dots & b_n & & \\ & \dots & \dots & \dots & \dots & \\ & & b_0 & b_1 & \dots & b_n \\ a_n & a_{n-1} & \dots & a_0 & & \\ & \dots & \dots & \dots & \dots & \\ & & a_n & a_{n-1} & \dots & a_0 \end{pmatrix}. \end{aligned} \tag{1.5}$$

What we understand by the transfinite resultant is actually the inverse limit of the latter meromorphic resultant. Indeed, one can check that for $n \geq 1$ the

Sylvester's determinants

$$\mathcal{R}(a,b)_n = a_0^{-n}b_0^{-n}\det\begin{pmatrix} b_0 & b_1 & \dots & b_n & & \\ & \dots & \dots & \dots & \dots & \\ & & b_0 & b_1 & \dots & b_n \\ a_n & a_{n-1} & \dots & a_0 & & \\ & \dots & \dots & \dots & \dots & \\ & & a_n & a_{n-1} & \dots & a_0 \end{pmatrix}$$
$$\equiv \mathcal{R}(\sum_{k=0}^{n} b_k z^{-k}, \sum_{k=0}^{n} a_k z^{k}) = \mathcal{R}(\sum_{k=0}^{n} a_k z^{k}, \sum_{k=0}^{n} b_k z^{-k})$$

satisfy the transfinite property (1.3): substitution $a_n = b_n = 0$ into the above determinant gives $\mathcal{R}(a,b)_{n-1}$ (recall also that the meromorphic resultant is a symmetric function of its arguments). This shows that the sequence $(\mathcal{R}(a,b)_n)_{n\geq 0}$ generates an element in $\mathbb{C}_\infty[a_0, a_0^{-1}, b_0, b_0^{-1}; a_1, b_1, \dots]$ denoted by $\mathcal{R}(a,b)$ and called the *transfinite resultant*. Its two initial approximants are:

$$\mathcal{R}(a,b)_1 = 1 - \frac{a_1b_1}{a_0b_0},$$
$$\mathcal{R}(a,b)_2 = 1 - \frac{a_1b_1 + 2a_2b_2}{a_0b_0} - \frac{a_0a_2b_1^2 + b_0a_1^2b_2 - a_2^2b_2^2 + a_1a_2b_1b_2}{a_0^2b_0^2}.$$

1.5. The transfinite elimination function

In many applications the so-called elimination function is more advantageous than the meromorphic resultant. It is defined by

$$\mathcal{E}_{f,g}(u,v) := \mathcal{R}(f-u, g-v),$$

where u and v are two independent complex variables. It is known (see [8]) that for any two meromorphic functions f and g on a closed Riemann surface this function is rational and satisfies the following elimination property:

$$\mathcal{E}_{f,g}(f(\zeta), g(\zeta)) \equiv 0.$$

The transfinite version of the elimination function is defined by

$$\mathcal{E}_{a,b}(u,v) = \lim_{\leftarrow n} \mathcal{R}(-u + \sum_{k=0}^{n} a_k z^{k+1}, -v + \sum_{k=0}^{n} b_k z^{-k-1}).$$

The latter transfinite function can be also viewed as a transfinite resultant with two distinguished null-variables u and v. The nth approximant is the determinant of size $(2n+2)\times(2n+2)$:

$$\mathcal{E}_{a,b}(u,v)_n = \frac{1}{(uv)^{-n-1}}\det\begin{pmatrix} v & -b_0 & \dots & -b_n & & \\ & \dots & \dots & \dots & \dots & \\ & & v & -b_0 & \dots & -b_n \\ -a_n & -a_{n-1} & \dots & u & & \\ & \dots & \dots & \dots & \dots & \\ & & -a_n & -a_{n-1} & \dots & u \end{pmatrix}. \quad (1.6)$$

We shall see (see Remark 2.2 below) that for $b = \bar{a}$, the transfinite elimination function is related to the exponential transform. In general the following analogue of elimination property holds.

Proposition 1.3. *Let* $f = \sum_{k=0}^{\infty} a_k z^{k+1}$ *and* $g = \sum_{k=0}^{\infty} b_k z^{-k-1}$. *Then*

$$\mathcal{E}_{a,b}(f, g) = 0,$$

in the sense that

$$\mathcal{E}_{a,b}(f_n, g_n)_n = 0, \quad n \geq 1, \tag{1.7}$$

where f_n *and* g_n *are the* n*th approximants of* f *and* g *respectively.*

Proof. It suffices to prove (1.7). To this aim notice that the resultant of the two polynomials

$$\widetilde{g}(z) = (g_n(z) - v)z^{n+1} = b_n + b_{n-1}z + \ldots + b_0 z^n - v z^{n+1},$$

and

$$\widetilde{f}(z) = f_n(z) - u = -u + a_0 z + a_1 z^2 + \ldots + a_n z^{n+1}$$

with respect to the variable z coincides with the determinant in (1.6):

$$\mathcal{E}_{a,b}(u, v)_n = \frac{1}{(uv)^{n+1}} \mathcal{R}_{\mathrm{pol}}(\widetilde{g}, \widetilde{f}).$$

By the characteristic property of the resultant, $\mathcal{E}_{a,b}(u, v)_n$ vanishes if and only if $\widetilde{g}$ and $\widetilde{f}$ have a common root $z_0 \in \mathbb{C}$, i.e. $\widetilde{f}(z_0) = \widetilde{g}(z_0) = 0$ for some complex z_0. This is equivalent to saying that $g_n(z_0) - v = 0$ and $f_n(z_0) - u = 0$. Hence $\mathcal{E}_{a,b}(f_n, g_n)_n$ equals identically zero which yields the desired property.

□

2. Transfinite functions on closed analytic curves

A general idea of a transfinite function on closed analytic curves is modeled on the following observation. Consider any parameterized curve $\Gamma = f_n(\mathbb{T})$, where f_n is the nth approximant to the transfinite series (1.4)

$$f_n(z) = \sum_{k=0}^{n} a_k z^{k+1}, \qquad a_0 > 0, \tag{2.1}$$

and $\mathbb{T}$ is the unit circle. Regarding the coefficients of $f_n(z)$ as a coordinate system on the space of parameterized curves Γ, many established functionals can be written as functions of the coefficients a_k. Those functionals which are polynomials in a_k for $k \geq 1$ will be called *admissible*.

Let $h : \Gamma \to \mathbb{C}$ be any admissible functional and h_n its resulting expression when Γ has the form (2.1). If the sequence (h_n) satisfies the condition (1.3), it extends to a transfinite function $\widetilde{h} \in \mathbb{C}_\infty[a_0, a_1, \bar{a}_1, \ldots]$ which is called the transfinite extension of h. In that case $\widetilde{h}$ can be thought of as the 'value' of h on the transfinite series $f(z) = \sum_{k=0}^{\infty} a_k z^{k+1}$ which, in its turn, can be regarded as an 'ideal' curve. Hence the formalism of transfinite functions can be applied to translate admissible functionals onto algebraic language. Below we demonstrate how this principle works with the Schwarz function as an example.

2.1. The Schwarz function and harmonic moments

We start with standard definitions. With any analytic curve Γ (not necessarily closed in general) one can associate the Schwarz function, that is a holomorphic in a neighborhood of Γ function $S(\zeta)$ such that

$$S(\zeta) = \bar{\zeta}, \quad \zeta \in \Gamma. \tag{2.2}$$

The above characteristic property is important when manipulating with the anti-holomorphic coordinate $\bar{\zeta}$ by substituting a *holomorphic* function $S(\zeta)$. The domain of definition of the Schwarz function is usually not a priori given, but one may always choose it to be symmetric with respect to Γ. Alternatively, one may think of $S(\zeta)$ only as a germ of an analytic function given on the curve.

In the other extreme, there is one case with a maximally unsymmetric domain of definition of the Schwarz function which singles out as being particularly tractable and having a rich theory: this is when $\Gamma = \partial\Omega$ for some domain Ω and the Schwarz function extends to being a meromorphic function in all of Ω. Then Ω is called quadrature domain, or algebraic domain [1]. It turns out that this requirement is rigid enough (see [1], [5]) to ensure that $S(\zeta)$ even is an algebraic function. Moreover, a simply connected quadrature domain is an image of the unit disk under a rational univalent function [1].

Now assume that Γ is a boundary of a simply connected domain Ω containing the origin and denote its Schwarz function by $S(\zeta)$. The following integrals make sense:

$$M_k(\Gamma) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \bar{\zeta}\zeta^k \, d\zeta, \qquad k \in \mathbb{Z}, \tag{2.3}$$

which for nonnegative k may be rewritten as

$$M_k(\Gamma) = \frac{\mathrm{i}}{2\pi} \iint_\Omega \zeta^k \, d\zeta \wedge d\bar{\zeta}, \qquad k \geq 0. \tag{2.4}$$

These quantities are known as harmonic or complex moments of the domain Ω. For negative k, (2.4) still make sense as principal value integrals. Alternatively, (2.3) may be turned into area integrals by passing to the complement $\mathbb{C} \setminus \overline{\Omega}$.

In what follows we shall think of the harmonic moments as generalized Laurent coefficients. More precisely, note that the substitution of the definition of $S(\zeta)$ into (2.3) yields

$$M_k(\Gamma) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma S(\zeta)\zeta^k \, d\zeta, \qquad k \in \mathbb{Z}, \tag{2.5}$$

hence, one can think of $M_{-k}(\Gamma)$ as the Laurent coefficients (with respect to Γ) of the shifted Schwarz function $\zeta S(\zeta)$. We shall write this as

$$S(\zeta) \sim \sum_{k\in\mathbb{Z}} M_k(\Gamma)\zeta^{-k-1}. \tag{2.6}$$

It worth to notice that (2.3), as well as (2.5), allows to define the moments for any parameterized curve. In a more generality, this reduces to considering the moments of a function rather than of a curve. Indeed, we recall that by the Riemann mapping theorem simply connected domains (with the origin inside) are in one-to-one correspondence with holomorphic and univalent in the unit disk $\mathbb{D}$ functions $f(z)$ normalized by $f(0) = 0$ and $f'(0) \in \mathbb{R}^+$.

Assume that the boundary of Ω is an analytic curve. Then a uniformizing function f may be chosen to be holomorphic in the closed unit disk. Hence substituting of $\zeta = f(z)$ into (2.3) gives

$$\mu_k(f) := M_k(f(\mathbb{T})) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma f^*(z) f^k(z) f'(z) dz, \qquad k \in \mathbb{Z}, \tag{2.7}$$

where $\mathbb{T} = \partial\mathbb{D}$ and a holomorphic in $\mathbb{C} \setminus \mathbb{D}$ function f^* is defined by

$$f^*(z) = \overline{f(1/\bar{z})}.$$

It is natural to refer to $\mu_k(f)$ as the moments of the function f.

Proposition 2.1. *Let $n \geq 1$ and let $f_n(z)$ be polynomial* (2.1) *such that $f_n(z)z^{-1}$ has no zeros in the closed unit disk. Then the $\mu_k(f_n)$ are rational functions of the coefficients of f_n. Moreover*

$$a_0^{n-2k-1}\mu_k(f_n) \in \mathbb{C}[a_0, a_1, \overline{a}_1, \dots, a_n, \overline{a}_n], \qquad \forall k \in \mathbb{Z}$$

and

$$\mu_k(f_n) = 0, \qquad \forall k \geq n+1 \tag{2.8}$$

In particular, $\mu_k(f_n)$ are admissible.

Proof. One can rewrite (2.7) as follows

$$\mu_k(f_n) = \mathrm{CT}_z(z f_n^* \, f_n' \cdot f_n^k),$$

where CT_z denotes constant term extraction (with respect to z). Then proposition follows from the fact that $z f' f_n^*$ contains only the terms z^m with $-n \leq i \leq n$ and from the Laurent expansion

$$f_n^k(z) = a_0^k z^k (1 + \frac{1}{a_0}\sum_{i=1}^n a_i z^i)^k = a_0^k z^k \sum_{j=0}^\infty \frac{k(k-1)\dots(k-j+1)}{j!} (\sum_{i=1}^n \frac{a_i z^i}{a_0})^j.$$

$\square$

It is a remarkable property of the harmonic moments that they can be extended as transfinite elements in the sense described in the beginning of this section. Indeed, a simple analysis of the above series for f_n^k shows that

$$\mu_k(f_n)|_{a_n = \overline{a}_n = 0} = \mu_k(f_{n-1}), \tag{2.9}$$

hence the following projective limit exists:

$$\mu_k(f) = \lim_{\leftarrow n} \mu_k(f_n) \in \mathbb{C}_\infty[a_0, a_1, \overline{a}_1, \dots]$$

and will be called the kth transfinite harmonic moment. The zero moment $\mu_0(f)$ is found to be

$$\mu_0 = a_0^2 + 2a_1\bar{a}_1 + 3a_2\bar{a}_2 + \dots,$$

and equals the normalized area of the domain $f_n(\mathbb{D})$ when f_n is a univalent function. Clearly μ_0 is positive and real (i.e., $\mu_0^* = \mu_0$).

The formulas for higher moments, especially for the negative ones, are much more involved. The first approximants for $k = 1$ and $k = -1$ are

$$\begin{aligned}
\mu_1(f_1) &= a_0^2\bar{a}_1,\\
\mu_1(f_2) &= a_0^2\bar{a}_1 + 3a_0a_1\bar{a}_2,\\
\mu_1(f_3) &= a_0^2\bar{a}_1 + 3a_0a_1\bar{a}_2 + 4a_0a_2\bar{a}_3 + 2a_1^2\bar{a}_3,
\end{aligned}$$

and

$$\begin{aligned}
\mu_{-1}(f_1) &= a_1 - \frac{a_1^2\bar{a}_1}{a_0^2},\\
\mu_{-1}(f_2) &= a_1 + \frac{2a_2\bar{a}_1}{a_0} - \frac{a_1^2\bar{a}_1 + 3a_2^2\bar{a}_2}{a_0^2} + \frac{a_1^3\bar{a}_2}{a_0^4}.
\end{aligned}$$

In general, for non-negative values of k Richardson's formula [17]

$$\mu_k(f_n) = \sum (s_0 + 1)a_{s_0}\cdots a_{s_k}\bar{a}_{s_0+\dots+s_k+k},$$

holds, where the summation is over all multiindices $(s_0, \dots, s_k)$, $0 \le s_j \le n$.

2.2. Harmonic moments via resultant

Here we describe briefly another way to obtain the harmonic moments with non-negative indices. Given a bounded domain Ω, the function of two complex variables defined by

$$\exp\Big[\frac{1}{2\pi \mathrm{i}}\int_\Omega \frac{d\zeta}{\zeta - z} \wedge \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{w}}\Big] =: E_\Omega(z, w) :\ (\mathbb{C} \setminus \overline{\Omega})^2 \to \mathbb{C}$$

is called the exponential transform of the domain Ω (see, e.g., [2], [16], [7]). Expanding the integral in power series in $1/z$ and $1/\bar{w}$ gives

$$E_\Omega(z, w) = 1 - \frac{1}{\bar{w}}C_\Omega(z) + \mathcal{O}\Big(\frac{1}{|w|^2}\Big) \tag{2.10}$$

as $|w| \to \infty$, with $z \in \mathbb{C} \setminus \overline{\Omega}$ fixed. Here

$$C_\Omega(z) = \frac{1}{2\pi \mathrm{i}}\int\limits_\Omega \frac{d\zeta \wedge d\bar{\zeta}}{z - \zeta} = \sum_{k\ge 0} \frac{M_k(\Gamma)}{z^{k+1}}, \quad \text{as } z \to \infty, \tag{2.11}$$

is the Cauchy transform of Ω, and $M_k(\Gamma)$ are defined as in (2.4).

When $\Omega = f_n(\mathbb{D})$, where f_n is a univalent in the closed unit disk polynomial (2.1), the sum in (2.11) contains only terms with degrees $k \le n$ and (2.10) becomes

$$E_{f_n(\mathbb{D})}(z, w) = 1 - \frac{1}{\bar{w}}\sum_{k=0}^{n} \frac{\mu_k(f_n)}{z^{k+1}} + \mathcal{O}\Big(\frac{1}{|w|^2}\Big).$$

On the other hand, the authors showed in [8] that the exponential transform of such $f_n(\mathbb{D})$ is the meromorphic resultant:

$$E_{f_n(\mathbb{D})}(z,w)=\mathcal{R}_\zeta(-z+\sum_{k=0}^{n}a_k\zeta^{k+1},-\bar{w}+\sum_{k=0}^{n}\bar{a}_k\zeta^{-k-1}) \tag{2.12}$$

Combing these formulas we obtain

$$\det\begin{pmatrix} 1 & & & -\frac{\bar{a}_n}{\bar{w}} & & \\ -\frac{a_0}{z} & \ddots & & \vdots & \ddots & \\ \vdots & & 1 & -\frac{\bar{a}_0}{\bar{w}} & & -\frac{\bar{a}_n}{\bar{w}} \\ -\frac{a_n}{z} & & -\frac{a_0}{z} & 1 & & \vdots \\ & \ddots & \vdots & & \ddots & -\frac{\bar{a}_0}{\bar{w}} \\ & & -\frac{a_n}{z} & & & 1 \end{pmatrix}=1-\frac{1}{\bar{w}}\sum_{k=0}^{n}\frac{\mu_k(f_n)}{z^{k+1}}+\mathcal{O}(\frac{1}{|w|^2}).$$

Hence the above determinant completely determines all the harmonic moments $\mu_k(f_n)$ for $0\le k\le n$ and expanding the determinant in $\bar{w}$, one gets explicit formulas.

Remark 2.2. Another corollary of (2.12) is that $E_{f_n(\mathbb{D})}(z,w)$ coincides with the nth approximant of the transfinite elimination function (1.6) for $a=(a_k)_{k\ge 0}$ and $b=(\bar{a}_k)_{k\ge 0}$. This can be thought as a transfinite analogue of the coincidence of the elimination function and the exponential transform on the level of transfinite functions.

2.3. The transfinite Schwarz function

By property (2.8) in Proposition 2.1 the series

$$S(f_n,\zeta)=\sum_{k\in\mathbb{Z}}\mu_k(f_n)\zeta^{-k-1}$$

contains only finite number of negative terms, hence may be interpreted as formal Laurent series with coefficients in $\mathbb{C}[a_0,a_1,\overline{a}_1,\dots,a_n,\overline{a}_n]$. It follows immediately from (2.9) that $S(f_n,\zeta)$ also satisfies the transfinity condition. Hence

$$S_f(\zeta)=\lim_{\leftarrow n}\sum_{k\in\mathbb{Z}}\mu_k(f_n)\zeta^{-k-1}$$

is well-defined on the level of formal Laurent series and in this setting formula (2.6) makes a rigorous sense. Moreover, the characteristic property (2.2) of the Schwarz function reads in the new notations as follows:

$$S_f(f)=f^*.$$

3. A generalized moment map

Let a closed Jordan analytic curve be given. The most natural choice of a coordinate system for an analytic curve is the coefficients of the uniformization map, which maps the unit disk onto the interior of the curve.

Another choice comes from the Schwarz function $S(\zeta)$ of the curve which by (2.2) contains complete information about the curve. Therefore the harmonic moments may be thought as coordinates. However one can extract information about the curve from harmonic moments in many different ways. We shall consider the following two. Let f_n is given by (2.1). Taking into account that μ_0 and a_0 are real (in fact, positive), we define the so-called *complete moment map* by

$$\mu(\bar{a}_n, \dots, \bar{a}_1, a_0, \dots, a_n) = (\mu_{-n}, \dots, \mu_0, \dots, \mu_n) : \mathbb{R} \times \mathbb{C}^{2n} \to \mathbb{R} \times \mathbb{C}^{2n}, \quad (3.1)$$

where $\mu_k = \mu_k(f_n)$ (recall that $\mu_k(f_n) = 0$ for $k \geq n+1$). It is well-known that the right half of the latter map (the moments with non-negative indices) determines f_n at least locally and usually one takes it as an alternative (to the coefficients a_k) set of coordinates (see, e.g., [10]).

Another moment map, which was treated recently in [12], [18], is a moment map consisting of the nonnegative moments and their conjugates:

$$\mu^*(\bar{a}_n, \dots, \bar{a}_1, a_0, \dots, a_n) = (\bar{\mu}_n, \dots, \bar{\mu}_1, \mu_0, \dots, \mu_n) : \mathbb{R} \times \mathbb{C}^{2n} \to \mathbb{R} \times \mathbb{C}^{2n}. \quad (3.2)$$

These two maps may be written in a common form

$$\phi(\bar{a}_n, \dots, \bar{a}_1, a_0, \dots, a_n) = (\phi_{-n}, \dots, \phi_{-1}, \phi_0, \dots, \phi_n), \quad (3.3)$$

with the generalized moments ϕ_k given by the integrals

$$\phi_k = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}} \Phi_k(f_n, f_n^*) f_n' dz = \mathrm{CT}_z(z f_n' \Phi_k(f_n, {f_n}^*)), \quad (3.4)$$

where $\Phi_k(\zeta, \bar{\zeta})$ are suitable functions. If $f : \mathbb{D} \to \Omega$ is a uniformizing map of a simply connected domain Ω then these moments are

$$\phi_k = \frac{1}{2\pi \mathrm{i}} \iint_{\Omega} \frac{\partial \Phi_k}{\partial \bar{\zeta}} \, d\zeta \wedge d\bar{\zeta} = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \Phi_k \, d\zeta.$$

For the complete moment map (3.1)

$$\Phi_k(\zeta, \bar{\zeta}) \equiv \zeta^k \bar{\zeta}, \quad (k \in \mathbb{Z})$$

and it is not hard to check that for (3.2)

$$\Phi_k(\zeta, \bar{\zeta}) = \begin{cases} \zeta^k \bar{\zeta} & \text{if } k \geq 0, \\ \frac{1}{1-k} \bar{\zeta}^{1-k} & \text{if } k \leq -1, \end{cases}$$

Our main result below shows that the Jacobian of the generalized moment map ϕ always splits into two distinguished factors: the first depends on a concrete form of the densities V_k and the second is the self-resultant of the derivative f'. To formulate it, it is convenient to set

$$a_{-n} = \bar{a}_n.$$

Theorem 3.1. *Let $n \geq 1$ and $\Phi_k(\zeta, \bar{\zeta})$, $-n \leq k \leq n$, be a system of rational functions. For $f_n(z) = z\sum_{k=0}^n a_k z^k$ introduce the residue matrix*

$$v_{kj} = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}} \Phi_k(f_n, {f_n}^*) \frac{dz}{z^{1+j}}. \tag{3.5}$$

Then the Jacobian of the generalized moment map (3.3) is

$$\frac{\partial \phi}{\partial a} \equiv \frac{\partial(\phi_{-n}, \ldots, \phi_0, \ldots, \phi_n)}{\partial(a_{-n}, \ldots, a_0, \ldots, a_n)} = 2a_0^{2n+1} \det(v_{kj}) \cdot \mathcal{R}(f_n', f_n'^*).$$

Here and in what follows we denote by (r_{kj}) the $(2n+1) \times (2n+1)$-matrix with entries r_{kj} with indices k, j running interval between $-n$ and n.

In the two cases discussed above, the corresponding matrices (v_{kj}) are easily found to be upper diagonal with a_0 to certain degrees on the diagonal. This yields the following corollary.

Corollary 3.2. *In the introduced above notation, we have for the Jacobians:*

$$\frac{\partial(\mu_{-n}, \ldots, \mu_0, \ldots, \mu_n)}{\partial(a_{-n}, \ldots, a_0, \ldots, a_n)} = 2a_0^{2n+1} \, \mathcal{R}(f_n', f_n'^*),$$

and

$$\frac{\partial(\bar{\mu}_n, \ldots, \bar{\mu}_1, \mu_0, \ldots, \mu_n)}{\partial(a_{-n}, \ldots, a_0, \ldots, a_n)} = 2a_0^{n^2+3n+1} \, \mathcal{R}(f_n', f_n'^*),$$

In particular, the transition Jacobian between these two maps is given by

$$\frac{\partial(\bar{\mu}_n, \ldots, \bar{\mu}_1, \mu_0, \ldots, \mu_n)}{\partial(\mu_{-n}, \ldots, \mu_{-1}, \mu_0, \ldots, \mu_n)} = a_0^{n^2+n}.$$

Proof of Theorem 3.1. Let $u(z)$ represent a direction for variation of $f_n(z)$ and let $d\phi_k(u)$ denote the directional derivative of ϕ_k taken at f_n along the function u. Then by virtue (3.4) we have

$$\begin{aligned} d\phi_k(u) &= \lim_{t \to 0} \frac{\phi_k(f + tu) - \phi_k(f)}{t} \\ &= \mathrm{CT}_z[zuf'\partial_\zeta \Phi_k(f, f^*) + zu^* f' \partial_{\bar{\zeta}} \Phi_k(f, f^*) + zu'\Phi_k(f, f^*)]. \end{aligned} \tag{3.6}$$

Integrating by parts we find for the last term in (3.6):

$$\begin{aligned} \mathrm{CT}_z[zu'\Phi_k(f, f^*)] &= \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}} \Phi_k(f, f^*) \, du \\ &= -\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}} (uf'\partial_\zeta \Phi_k(f, f^*) + uf^{*\prime}\partial_{\bar{\zeta}} \Phi_k(f, f^*))dz \\ &= \mathrm{CT}_z\Big[-zuf'\partial_\zeta \Phi_k(f, f^*) + \frac{1}{z} uf'^* \partial_{\bar{\zeta}} \Phi_k(f, f^*)\Big], \end{aligned}$$

where we used

$$(f^*(z))' = -\frac{1}{z^2} f'^*(z).$$

Substitution of this into (3.6) gives

$$d\phi_k(u) = \mathrm{CT}_z[(zf'u^* + \frac{1}{z}f'^*u) \cdot \partial_{\bar\zeta}\Phi_k(f, f^*)],$$

and setting $h(z) = \frac{u(z)}{z} = h_0 + h_1 z + \ldots + h_n z^n$ we arrive at the following formula:

$$d\phi_k(zh) = \mathrm{CT}_z[(f'h^* + f'^*h) \cdot \partial_{\bar\zeta}\Phi_k(f, f^*)] \equiv \sum_{j=-n}^{n} \phi_{kj} h_j, \tag{3.7}$$

where $h_{-k} = \bar h_k$, $k \geq 1$. In this notation the required Jacobian spells out as

$$\frac{\partial\phi}{\partial a} = \det(\phi_{kj})_{-n\leq k,j\leq n}. \tag{3.8}$$

Using (3.5) we find from (3.7)

$$\sum_{j=-n}^{n} \phi_{kj} h_j = \sum_{i,j=-n}^{n} v_{ki} u_{ij} h_j$$

where

$$\mathrm{CT}_z[(f'h^* + f'^*h)z^i] = \sum_{j=-n}^{n} u_{ij} h_j. \tag{3.9}$$

This yields by virtue of (3.8)

$$\frac{\partial\phi}{\partial a} = \det(v_{ki}) \cdot \det(u_{ij}).$$

Hence we only need to find the determinant of $U = (u_{jm})$. Let us write

$$f'(z) = \sum_{k=0}^{n} b_k z^k, \qquad f'^*(z) = \sum_{k=0}^{n} b_{-k} z^{-k},$$

where $b_k = (k+1)a_k$ and $b_{-k} = (k+1)\bar a_k$ for $k \geq 0$ (notice that this index notation is consistent with $b_0 = a_0 \in \mathbb{R}$). Then an explicit form of the matrix U is easily found from (3.9):

$$U = \begin{pmatrix}
b_0 & & & & b_{-n} & & & & \\
b_1 & b_0 & & & b_{1-n} & b_{-n} & & & \\
\vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & & \\
b_{n-1} & b_{n-2} & \dots & b_0 & b_{-1} & b_{-2} & \dots & q_{n-1} & \\
b_n & b_{n-1} & \dots & b_1 & 2b_0 & b_{-1} & \dots & b_{1-n} & b_{-n} \\
 & b_n & \dots & b_2 & b_1 & b_0 & \dots & b_{1-n} & b_{1-n} \\
 & & \ddots & \vdots & \vdots & & \ddots & \vdots & \vdots \\
 & & & b_n & b_{n-1} & & & b_0 & b_{-1} \\
 & & & & b_n & & & & b_0
\end{pmatrix}.$$

Denote by U_k the kth column $(u_{ik})_{i=-n}^{n}$ in U. Then

$$b_0 U_0 + \sum_{i=1}^{n} b_{-i} U_{-i} - \sum_{i=1}^{n} b_i U_i = 2b_0 Z, \tag{3.10}$$

where the column vector Z has the form

$$Z = (b_{-n}, \dots, b_{-1}, b_0, 0, \dots, 0)^\top$$

with the last n entries equal to zero. It follows then from (3.10) that

$$\det U = 2\det \begin{pmatrix} b_0 & & & & b_{-n} & & & & \\ b_1 & b_0 & & & b_{1-n} & b_{-n} & & & \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & & \\ b_{n-1} & b_{n-2} & \dots & b_0 & b_{-1} & b_{-2} & \dots & b_{-n} & \\ b_n & b_{n-1} & \dots & b_1 & b_0 & b_{-1} & \dots & b_{1-n} & b_{-n} \\ & b_n & \dots & b_2 & & b_0 & \dots & b_{2-n} & b_{1-n} \\ & & \ddots & \vdots & & & \ddots & \vdots & \vdots \\ & & & b_n & & & & b_0 & b_{-1} \\ & & & & & & & & b_0 \end{pmatrix}$$

Now expanding the latter determinant by the last row and taking into account (1.5), we get

$$\det U = 2b_0^{2n+1}\, \mathcal{R}(\sum_{k=0}^{n} b_k z^k, \sum_{k=0}^{n} b_{-k} z^{-k}) = 2a_0^{2n+1}\, \mathcal{R}(f', f'^*),$$

which finishes the proof. $\Box$

References

[1] D. Aharonov and H. S. Shapiro, Domains in which analytic functions satisfy quadrature identities, *J. Analyse Math.* **30** (1976), 39–73.

[2] R.W. Carey and J.D. Pincus, An exponential formula for determining functions, *Indiana Univ. Math. J.* **23**(1974), 1031–1042

[3] L. A. Galin, Unsteady filtration with a free surface, *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **47**.

[4] I.M. Gelfand, M.M. Kapranov and A.V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants.* Birkhäuser Boston, Inc., Boston, MA, 1994.

[5] B. Gustafsson, Quadrature identities and the Schottky double, *Acta Appl. Math.*, **1** (1983), 209–240.

[6] B. Gustafsson, On a differential equation arising in a Hele-Shaw flow moving boundary problem, Ark. Mat. **22** (1984), 251–268.

[7] B. Gustafsson and M. Putinar, An exponential transform and regularity of free boundaries, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4), Vol. **XXVI**(1998), 507–543.

[8] B. Gustafsson and V. Tkachev, The resultant on compact Riemann surfaces, *Comm. Math. Phys.*, to appear.

[9] B. Gustafsson, A. Vasil'ev, *Conformal and Potential Analysis in Hele-Shaw Cells*, Birkhauser Verlag, 2006.

[10] I.K. Kostov, I. Krichever, M. Mineev-Weinstein, P. Wiegmann and A. Zabrodin, The τ-function for analytic curves. Random matrix models and their applications, 285–299, *Math. Sci. Res. Inst. Publ.*, **40**, Cambridge Univ. Press, Cambridge, 2001.

[11] I. Krichever, M. Mineev-Weinstein, P. Wiegmann and A. Zabrodin, Laplacian growth and Whitham equations of soliton theory. *Phys. D*, **198**(2004), no. 1-2, 1–28.

[12] O. Kuznetsova and V. Tkachev, Ullemar's formula for the Jacobian of the complex moment mapping, *Complex Variables and Applications*, **49**(2004), No 1, 55–72.

[13] I. D. Macdonald, *Symmetric functions and Hall polynomials*, 2nd Ed., Oxford Math. Monographs, 1995.

[14] M. Mineev-Weinstein and A. Zabrodin, Whitham-Toda Hierarchy in the Laplacian Growth Problem. *J. Nonlin. Math. Phys.*, **8**(2001), 212–218.

[15] P.J. Polubarinova-Kotschina, *On the Displacement of the Oilbearing Contour. C. R. (Dokl.) Acad. Sci. URSS, n. Ser.* **47**, 1945, 250–254.

[16] M. Putinar, Extremal solutions of the two-dimensional L-problem of moments, II, *J. Approx. Th.* **92** (1998), 38–58

[17] S. Richardson, Hele-Shaw flows with a free boundary produced by the injection of fluid into a narrow channel. *J. Fluid Mech.*, **56**(1972), 609–618.

[18] V. Tkachev, Ullemar's formula for the moment map, II, *Linear Algebra Appl.*, **404** (2005), 380-388

[19] C. Ullemar, Uniqueness theorem for domains satisfying quadrature identity for analytic functions. *TRITA-MAT 1980-37, Mathematics.*, (1980) Preprint of Royal Inst. of Technology, Stockholm.

[20] van der Waerden, *Algebra* I, Springer-Verlag.

[21] P.B. Wiegmann and A. Zabrodin, Conformal maps and dispersionless integrable hierarchies, *Commun.Math.Phys.* **213** (2000) 523-538

Björn Gustafsson
Mathematical department, KTH
e-mail: `gbjorn@kth.se`

Vladimir Tkachev
Volgograd State University/ KTH
e-mail: `tkatchev@kth.se`