Life-time of minimal tubes and coefficients of univalent functions in a circular ring

Vladimir G. Tkachev

Abstract. We obtain various estimates of the life-time of two-dimensional minimal tubes in $\mathbb{R}^3$ by potential theory methods.

1. Introduction.

Let $x = (x_1, x_2, \ldots, x_n, x_{n+1})$ be a point in Euclidean space $\mathbb{R}^{n+1}$ with the *time* axis Ox_{n+1} and M be a p-dimensional Riemannian manifold, $2 \le p \le n$.

Definition 1. We say that a surface $\mathcal{M} = (M, \mathbf{u})$ given by C^2-immersion $\mathbf{u} : M \to \mathbb{R}^{n+1}$ is a *tube* with the projection interval $\tau(\mathcal{M}) \subset Ox_{n+1}$, if (i) for any $\tau \in \tau(\mathcal{M})$ the sections $\Sigma_\tau = f(\mathcal{M}) \cap \Pi_\tau$ by hyperplanes $\Pi_\tau = \{x \in \mathbb{R}^{n+1}_1 : x_{n+1} = \tau\}$ are not empty compact sets; (ii) for $\tau', \tau'' \in \tau(\mathcal{M})$ any part of $\mathcal{M}$ situated between two different $\Pi_{\tau'}$ and $\Pi_{\tau''}$ is a compact set.

Definition 2. A surface $\mathcal{M}$ is called *minimal* if the mean curvature of $\mathcal{M}$ vanishes everywhere.

It is the well known fact (see [5], p.331) that the minimality condition of $\mathcal{M}$ is equivalent to that all coordination functions of the immersion $\mathbf{u}$ are harmonic. For this reason, the two-dimensional minimal tubes can be considered as direct analog of the closed relative string conception in the modern nuclear physics (cf. [2]). This approach was proposed by V.M.Miklyukov and the author in [7] for an arbitrary dimension p.

From this point of view many intrinsic geometric invariants of $\mathcal{M}$ have the natural physical meaning. Namely, the length of the projection interval $|\tau(\mathcal{M})|$ can be interpreted as a *life-time* of the tube $\mathcal{M}$.

To introduce the following important characteristic we denote by ν the unit normal to Σ_τ with respect to $\mathcal{M}$ which is co-directed with the time-axis Ox_{n+1}. Then by virtue of the harmonicity of the coordinate functions $u_k(m) = x_k \circ \mathbf{u}(m)$,

---

1991 *Mathematics Subject Classification.* Primary 53A10; Secondary 53A35.

This paper was supported in part by Russian Fundamental Researches Fund, project 93-011-176

Published in: *Bull. Soc. Sci. Lett. Łódź. Sér. Rech. Déform.* **20** (1995), 19–26.

$1 \le k \le n+1$, the flow integrals

$$J_k = \int_{\Sigma_\tau} \langle \nabla u_k, \nu \rangle \, d\Sigma$$

are independent of $\tau \in \tau(\mathcal{M})$. Here $d\Sigma$ is the 1-Hausdorff measure along Σ_τ.

Definition 3. We call $Q(\mathcal{M}) = (J_1, J_2, \ldots, J_{n+1}) \in \mathbb{R}^{n+1}$ the *full flow-vector* of $\mathcal{M}$.

We notice the positiveness of J_{n+1} as a consequence of the choice of ν direction. Moreover, $Q(\mathcal{M})$ is an 1-homogeneous functional of $\mathcal{M}$ under the homotheties group action in $\mathbb{R}^{n+1}$. Let us denote by $\alpha(\mathcal{M})$ the angle between $Q(\mathcal{M})$ and the time-axis Ox_{n+1}.

In this paper we are interested in the following question: What sufficient conditions yield the finiteness of the time-life of a two-dimensional minimal tube? As it shown in the series of papers [6]–[8], in the case $p \ge 3$ this quantity is always finite and the following estimation holds

$$|\tau(\mathcal{M})| \le \varrho(\mathcal{M}) c_p,$$

where c_p depends only on p, and $\varrho(\mathcal{M})$ is the smallest diameter of sections Σ_τ. The last relationship is sharp and the equality occurs if and only if $\mathcal{M}$ is a minimal surface of revolution.

A special feature of the two-dimensional case is that there exist tubes with finite as well as infinite values of the life-time. Indeed, a family of slanted minimal surfaces with circular cross-sections Σ_τ was discovered by B. Riemann [10]. Some other recent examples can also be found in [4].

In this paper we prove

Theorem 1. *Let $\mathcal{M}$, $\dim\mathcal{M} = 2$ be a minimal two-connected tube with univalent Gaussian mapping. If the angle $\alpha(\mathcal{M})$ is different from zero, then the life-time $|\tau(\mathcal{M})|$ of $\mathcal{M}$ is finite and*

$$\tau(\mathcal{M}) \le \frac{\pi \|Q\| \cos\alpha(\mathcal{M})}{\ln\tan(\frac{\pi}{4} + \frac{\alpha}{2})}.$$

Let us denote by $a_0[f]$ the central coefficient of the Laurent decomposition of an holomorphic function $f(z)$ in an annulus $K_R = \{z : 1/R < |z| < R\}$, i.e.

$$a_0[f] \equiv \int_{C_1} \frac{g(\zeta)\, d\zeta}{\zeta}.$$

where C_1 is the unite circle $\{z \in \mathbb{C} : |z| = 1\}$. The following auxiliary assertion is a key ingredient in the proof of Theorem 1.

Theorem 2. *Let $g(z)$ be a univalent holomorphic function defined in the annulus K_R omitting zero. Assume that*

$$a_0[g] = \lambda, \qquad a_0[1/g] = -\lambda, \tag{1}$$

for some real positive λ. *Then*

$$\ln R \le \ln R_0(\lambda) = \frac{\pi^2}{\ln(\lambda + \sqrt{1+\lambda^2})} \tag{2}$$

Remark 1. We note that estimate (2) has well asymptotic behaviour for $R \to \infty$ as shows Riemannian example mentioned above. But we cannot now present the sharp value of $R_0(\lambda)$. Nevertheless, it seemed us very probably that the following conjecture is true.

Remark 2. The best upper bound in the left side of (2) is achieved for holomorphic function $g_0(z)$ which provides a conformal map of the annulus K_R onto the plain $\mathbb{C}$ with two slits: $(-1/\alpha; 0)$ and $(\alpha; +\infty)$, for the suitable choice of parameter α.

Acknowledgement. I wish to thank V.M.Miklyukov for many useful discussions concerning the topic of this paper.

2. Proof of Theorem 2

Let $\Gamma = \{C_\rho : 1/R < \rho < R\}$ be a family of all concentric circles $C_\rho = \{z : |z| = \rho\}$ in the annulus K_R. It follows easily from the non-vanishing property of $g(z)$ that the loop C_1 in the integrals (1) may be replaced by an arbitrary circle $C_\rho \in \Gamma$. It follows from the mean value theorem and (1) that for every $\rho \in (1/R; R)$ there exist t_1 and t_2 such that

$$\text{Re}\, g(\rho e^{it_1}) = \lambda \quad \text{and} \quad \text{Re}\, \frac{1}{g(\rho e^{it_2})} = -\lambda. \tag{3}$$

Let $\gamma_\rho = g(C_\rho)$. Then by virtue of the univalence of $g(z)$, the curve γ_ρ is the simple Jordan one. Let $g(\rho e^{it}) = x(t) + iy(t)$ be the representation of γ_ρ. Then we obtain from (3)

$$x(t_1) = \lambda; \qquad x^2(t_2) + y^2(t_2) + \frac{1}{\lambda}x(t_2) = 0.$$

The last relations have the helpful geometric interpretation:

($\star$) The curve γ_ρ has a non-empty intersection with the vertical rightline $L_1 = \{z : \text{Re}z = \lambda\}$ and the circle $L_2 = \{z : |z + 1/2\lambda| = 1/2\lambda\}$.

We shall make use the technique from the potential theory (the length-are method). Recall the exact definition. Let E be a family of locally rectifiable curves γ and $\varphi(z) \ge 0$ be a Baire function with the property

$$\int_\gamma \varphi(z)\,|dz| \ge 1,$$

for every $\gamma \in E$. The infimum

$$\text{mod } E = \inf \int \varphi^2(z)\,dx\,dy$$

over all such $\varphi(z)$ is called a *conformal module* of the family E.

Then it is known (see [1]) that mod E is the conformal invariant. As a consequence we obtain in our situation

$$\operatorname{mod}\Gamma = \operatorname{mod}\Gamma_1, \tag{4}$$

where $\Gamma_1 = \{\gamma_\rho : 1/R < \rho < R\}$.

Let us denote by D the two-dimensional domain

$$D = \left\{ z : \operatorname{Re} z < \lambda;\ \left| z + \frac{1}{2\lambda} \right| > \frac{1}{2\lambda} \right\}.$$

Using the $(\star)$-property, we can find for every $\rho \in (1/R; R)$ the continuum $\gamma'_\rho \subset \gamma_\rho$ joining the boundary components of D. Then a family Γ_2 consisting of all continua γ'_ρ is "shorter" than Γ_1 and it follows from Theorem 1.2, [1] that

$$\operatorname{mod}\Gamma_1 \le \operatorname{mod}\Gamma_2. \tag{5}$$

On the other hand, Γ_2 is the subfamily of $\Gamma(D)$, where the last term means the family of *all* curves joining the boundary components of a domain D. The monotonicity property of infimum and Definition 4 lead to the following inequality

$$\operatorname{mod}\Gamma_2 \le \operatorname{mod}\Gamma(D). \tag{6}$$

Now, combining the standard fact

$$\operatorname{mod}\Gamma = \frac{\ln R}{\pi} \tag{7}$$

with relations (4), (5) and (6) we arrive at the following inequality

$$\frac{\ln R}{\pi} \le \operatorname{mod}\Gamma(D).$$

To compute the last module we note that the linear-fractional function

$$f(z) = \frac{1}{\lambda^*} \cdot \frac{z + \lambda^*}{1 - z\lambda^*}$$

maps D onto an annulus $K_1 = \{w : 1 < |w| < 1/{\lambda^*}^2\}$, where $\lambda^* = \sqrt{\lambda^2 + 1} - \lambda$. Thus, using the invariance property of conformal module we obtain

$$\frac{\ln R}{\pi} \le \operatorname{mod}(D) \equiv \frac{2\pi}{\ln(1/{\lambda^*}^2)} = \frac{\pi}{\ln(\lambda + \sqrt{1 + \lambda^2})}.$$

and Theorem 2 is proved.

3. The Gaussian map of two-dimensional minimal tubes and their full-flow vector

In this section we express the full flow-vector of an arbitrary two-dimensional tube $\mathcal{M} \in \mathbb{R}^n$ via Chern-Weierstrass representation for minimal surfaces. Namely, if $\mathcal{M}$ is a two-connected surface then we can arrange that $\mathcal{M}$ is conformally equivalent to an annulus K_R for the appropriate $R > 1$. Then there exist the corresponding parametrization of $\mathcal{M}$ (see [9]):

$$\mathbf{u}(z) = \mathrm{Re} \int_{z_0}^{z} F(\zeta)\, d\zeta \ : K_R \to \mathbb{R}^n,$$

where

$$F(z) = (\varphi_1(\zeta), \ldots, \varphi_n(\zeta))$$

and $\varphi_i(\zeta)$ are holomorphic functions satisfying the following conditions

$$\sum_{i=1}^{n} \varphi_i(\zeta)^2 = 0; \tag{8}$$

and

$$\mathrm{Re} \int_{|z|=1} F(\zeta)\, d\zeta = \mathbf{0}. \tag{9}$$

Lemma 1. *Under the above hypotheses we have*

$$Q(\mathcal{M}) = \mathrm{Im} \int_{|z|=1} F(\zeta)\, d\zeta. \tag{10}$$

Proof. It sufficient to show that

$$J_k \equiv \int_{\Sigma_\tau} \langle \nabla u_k, \nu \rangle \, d\Sigma = \mathrm{Im} \int_{|z|=1} \varphi_k(\zeta)\, d\zeta, \tag{11}$$

for every $k = 1, 2, \ldots, n+1$.

To prove (11) we introduce the conjugate to $u_k(z)$ function $v_k(z)$ by

$$v_k^*(z) = \mathrm{Im} \int_{z_0}^{z} \varphi_k(\zeta)\, d\zeta,$$

We notice that $v_k(z)$ in general is a multivalued function. On the other hand, the covariant derivative ∇v_k is well defined and using the properties of Hodge $\star$ - operator we have

$$\int_{\Sigma_\tau} \langle \nabla u_k, \nu \rangle \, d\Sigma = \int_{\Sigma_\tau} \langle \star\nabla u_k, \star\nu \rangle \, d\Sigma = \int_{\Sigma_\tau} \langle \nabla v_k, \star\nu \rangle \, d\Sigma =$$

$$= \int_{\Sigma_\tau} d\, v_k = \mathrm{Im} \int_{|z|=1} \varphi_k(\zeta)\, d\zeta,$$

and (11) is proved. □

In our case $n = 2$, Chern-Weierstrass representation can be simplified in the following classic way. Namely, there exist a holomorphic function $f(z)$ and a meromorphic function $g(z)$ which are well defined in the annulus K_R and such that

$$F(z) = \left((1-g^2)f;\ i(1+g^2)f;\ 2gf\right). \tag{12}$$

Moreover, poles of $g(z)$ coincide with zeros of $f(z)$ and the order of a pole of $g(z)$ is precisely the order of the corresponding zero of $f(z)$. We emphasize that $g(z)$ is a composition of the stereographic projection and Gaussian map of $\mathcal{M}$.

Lemma 2. *In our assumptions*

$$2fg \equiv \frac{\langle Q(\mathcal{M}), e_3\rangle}{2\pi z}, \tag{13}$$

and $g(z)$ omits the zero and infinity values.

Proof. We use the method proposed by M. Schiffman in [11]. We recall that the coordinate function $u_3(z)$ is harmonic in the annulus K_R and by virtue of Definition 1,

$$\lim_{z\to 1/R} u_3(z) = \tau_1, \qquad \lim_{z\to R} u_3(z) = \tau_2, \tag{14}$$

where $\tau(\mathcal{M}) = (\tau_1; \tau_2)$ is the projection of the tube $\mathcal{M}$ onto x_3-axis.

We consider an auxiliary harmonic function

$$h(z) = \tau_1 + \frac{\tau_2 - \tau_1}{2\ln R} \ln|z|.$$

It is easily seen that $h(z)$ satisfies (14). Thus $h_1(z) = u_3(z) - h(z)$ is harmonic in the annulus and

$$\lim_{z\to\partial K_R} h_1(z) = 0.$$

Then the maximum principle implies that $h_1(z) \equiv 0$ everywhere in K_R and hence

$$u_3(z) \equiv \tau_1 + \frac{\tau_2 - \tau_1}{2\ln R} \ln|z|. \tag{15}$$

In particular, it follows from (15) that

$$du_3(z) \equiv \frac{\tau_2 - \tau_1}{\ln R} \cdot \frac{z}{|z|^2}$$

doesn't vanish in K_R. We have, as a consequence, the normal $n(z)$ to $\mathcal{M}$ isn't parallel to e_3 at any point. Taking into account the above remark about the geometrical sense of $g(z)$ we obtain that $g(z) : K_R \to \mathbb{C} - \{0; \infty\}$.

By comparing of (15) and (12) we deduce that

$$2g(z)f(z) = \frac{\tau_2 - \tau_1}{2\ln R} \cdot \frac{dz}{z}. \tag{16}$$

In order to eliminate $\ln R$ from the latter equality we substitute (16) into (12), and after using (10) we obtain

$$\ln R = \frac{\pi(\tau_2 - \tau_1)}{J_3}. \tag{17}$$

On substituting of the found relationship into (16) we arrive at the conclusion of the lemma. □

4. Proof of Theorem 1

Let us denote $w = (J_1 + iJ_2)/J_3$. Combining Lemma 2, (12) and (9) we obtain

$$\int_{C_1} \frac{1 - g^2(\zeta)}{2g(\zeta)} \frac{d\zeta}{\zeta} = 2\pi w_1 i,$$

$$\int_{C_1} \frac{1 + g^2(\zeta)}{2g(\zeta)} \frac{d\zeta}{\zeta} = 2\pi w_2.$$

Simplifying the last expressions and denoting $w = |w| \cdot e^{i\theta}$, $g_1(z) = -e^{-i\theta} g(z)$ give the following system

$$\frac{1}{2\pi} \int_{C_1} \frac{g_1(\zeta) d\zeta}{\zeta} = |w|,$$

$$\frac{1}{2\pi} \int_{C_1} \frac{d\zeta}{g_1(\zeta)\zeta} = -|w|.$$

Applying Theorem 2 we arrive at the inequality

$$\ln R \leq \frac{\pi^2}{|w| + \sqrt{1 + |w|^2}}$$

where $|w| \equiv |J_1+iJ_2|/|J_3| = \tan \alpha(\mathcal{M})$. Using (17) we obtain the required estimate and the theorem is proved.

References

[1] L. Ahlfors, *Lectures on quasiconformal mappings*, New York, Van Nostrand Math. Studies, 1966.

[2] L. Brink and M. Henneaux, *Principles of string theory*, Plenum Press, New York, 1988.

[3] M. Callahan, D. Hoffman and W. H. Meeks, *Embedded minimal surfaces with an infinite number of ends*, Invent. math., 96(1989), 459-505.

[4] D. Hoffman and W. H. Meeks, *Minimal surfaces based on the catenoid*, Amer. Math. Monthly, 97(1990), 702-730.

[5] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. 2, Intersci. Publish., New York, 1969.

[6] V. A. Klyachin, *Estimate of spread for minimal surfaces of arbitrary codimension*, Sibirsk. Mat. Zh., 33(1992), N 5, p.201-207.

[7] V. M. Miklyukov and V. G. Tkachev, *Some properties of tubular minimal surfaces of arbitrary codimension*, Math. USSR-Sb., 180(1989), 1278-1295.

[8] V. M. Miklyukov and V. D. Vedenyapin, *Extrinic dimension of tubular minimal hypersurfaces*, Math. USSR-Sb., 131(1986), 240-250.

[9] R. Osserman, *A survey of minimal surfaces*, New York: Dover Publications, 1987.

[10] B. Riemann, *Qeuvres Mathematiques de Riemann*, Paris, Gautheriers-Villars, 1898.

[11] M. Schiffman, *On surfaces of stationary area bunded of two circles, or convex curves, in parallel planes*, Annals of Math., 63(1956), 77-90.