Journal of Mathematical Sciences, Vol. 72, No. 4, 1994

SOME ESTIMATES OF THE MEAN CURVATURE OF NONPARAMETRIC SURFACES GIVEN OVER DOMAINS IN R^n

V. G. Tkachev

UDC 517.95

Questions about the behavior of the mean curvature of surfaces given in the form of graph $X_{n+1} = f(x)$ over an arbitrary domain Ω in $\mathbb{R}^n$ are considered. It is proved, for example, that if mean curvature H is a continuously monotonically increasing function of coordinates x_{n+1} in $\mathbb{R}^{n+1}$, then the following assertions are fulfilled: a) if $\Omega = \mathbb{R}^n$, then $H = 0$, that is, the graph is a minimal surface; b) if $\partial\Omega \neq \phi$, then

$$\sup_{x\in\Omega} |H(f(x))|\cdot \mathrm{dist}(x;\,\partial\Omega) \leq 1 \qquad (*)$$

is true. Different special cases of Ω are considered, for which exact values of the constant on the right-hand side of (*) are obtained.

1. In this paper we discuss properties of the mean curvature of the class of nonparametrized surfaces introduced below, which includes, for example, surfaces of constant mean curvature and surfaces satisfying the capillarity equation.

In what follows we consider only oriented surfaces given as graphs over a fixed domain $\Omega \subset R^n$. The direction of the normal vector to the surface is chosen to be consistent with the positive direction of the $(n + 1)$ st coordinate in ambient space R^{n+1}. We denote by $x_1, x_2, \ldots, x_{n+1}$ a standard collection of coordinate functions in $(n + 1)$-dimensional Euclidean space R^{n+1} and agree to regard R^n as a hyperplane in R^{n+1} defined by $x_{n+1} = 0$. Everywhere below, unless otherwise specified, by $H-H(t)$ we mean a continuous decreasing function. Let us define a surface $\mathscr{F}$ as the graph of a C^2-solution $f(x) = f(x_1, \ldots, x_n)$ of

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\Big/\sqrt{1+|\nabla f|^2}\right) = nH(f(x)) \qquad (1)$$

in Ω. It is well known that $H(f(x))$ determines the mean curvature of $\mathscr{F}$ at the point $x \in \Omega$.

Let $\bar{\Omega}$ denote the closure of Ω in R^n and let P, Q be nonempty subsets in Ω, $P \cap Q = \phi$.

For a fixed $\alpha \geq 1$ we introduce the α-capacitance of capacitor $(P, Q; \Omega)$ (see [1]) letting

$$\mathrm{cap}_\alpha(P, Q; \Omega) = \inf\int_\Omega |\nabla\varphi|^\alpha dx, \quad dx = dx_1\ldots dx_n, \qquad (2)$$

where the greatest lower bound is taken over all finite functions $\varphi(x):\Omega \to [0; 1]$ that are locally Lipschitz in $\bar{\Omega}$, continuous in Ω, and are equal to zero on Q and one on P, respectively. We say [2, p.59], that a compact set P has zero capacitance if there exists a closed set Q and $Q \cap P = \phi$ such that complement $R^n \backslash Q$ is a bounded set and $\mathrm{cap}_n(P, Q; R^n) = 0$ is fulfilled. The closed set $P \subset R^n$ has capacitance zero if its every compact subset is like that.

The following estimate for the integral mean curvature of the introduced class of surfaces is true.

THEOREM 1. Suppose that P is a compact subset of Ω and $f(x)$ is a solution of (1) in Ω. Then for any $\alpha \geq 2$

Translated from Ukrainskii Geometricheskii Sbornik, No. 35, pp. 135-150, 1992. Original article submitted October 31, 1990.

1072-3374/94/7204-3250$12.50

$$\int_P |H(f(x))|^{\alpha} dx \le \left(\frac{\alpha}{n}\right)^{\alpha} \mathrm{cap}_{\alpha}(P, \partial Q; \Omega). \tag{3}$$

COROLLARY 1. Let $f(x)$ be a solution of (1) determined everywhere in R^n except, possibly, a closed set P of zero capacitance. Then the mean curvature $H(f(x)) \equiv 0$ everywhere in $R^n \backslash P$. In particular, if

a) $2 \le n \le 7$, then $f(x)$ is a linear function,

b) $H(t)$ is strictly monotonic, then $f(x) \equiv$ const for any n.

In the two-dimensional case, when $H(t)$ is sign constant, $H'(t) \ge 0$, and P is empty, Assertion a) was proved by Cheng and Yau in [3] using a method based on estimating the rate of the increase of the volume of a geodesic ball on Riemannian manifolds.

We preface the proof of Theorem 1 with an auxiliary assertion.

LEMMA 1. Let $H(t){:}R \to R$ be a continuous monotonic function. Then for any ε there exists a function $H_\varepsilon(t) \in C^\infty(R)$, $H_\varepsilon'(t) \ge 0$ such that

$$\begin{aligned} &(\mathrm{i}) \quad |H_\varepsilon(t)| \le |H(t)|; \\ &(\mathrm{ii}) \quad |H_\varepsilon(t) - H(t)| \le \varepsilon, \ \forall t \in R. \end{aligned} \tag{4}$$

Proof. It is not hard to note that the general case is easily reduced to $H(t)$ specified in $[0, +\infty]$, $H(0) = 0$ and $H(t) > 0$ for $t > 0$. We can assume that $H(t)$ is not bounded as $t \to +\infty$ (otherwise our line of argument changes insignificantly). Consider a sequence of points $\alpha_k \in R$ given by

$$\alpha_k = \max_{H(t) \le \frac{\varepsilon k}{2}} \{t\}; \qquad \alpha_0 = 0.$$

Clearly, α_k strictly increases and all the α_k are finite. Let

$$\sigma(t) = \begin{cases} \exp(1 - t^{-2}); 0 < t \le 1; \\ 0, \ t = 0. \end{cases}$$

It is easy to see that $\sigma(t) \in C^\infty[0; 1]$. The desired function $H_\varepsilon(t)$ assumes the form

$$H_\varepsilon(t) = \begin{cases} 0, 0 \le t \le \alpha_1 \\ H(\alpha_i) + \frac{\varepsilon}{2}\sigma\left(\frac{t - \alpha_i}{\alpha_{i+1} - \alpha_i}\right), \ t \in [\alpha_i; \alpha_{i+1}]. \end{cases}$$

Let us return to the proof of Theorem 1. Let $\varepsilon > 0$ and suppose that $H_\varepsilon(t)$ is the corresponding approximation of $H(t)$ satisfying (4). Let us fix the function $\varphi = \varphi(x)$ admissible in (2) for calculating the capacitance of $(P, \partial\Omega; \Omega)$. From (1) it follows that

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\varphi^\alpha(x) H_\varepsilon^{(\alpha-1)}(f(x)) \frac{f x_i}{\sqrt{1+|\nabla f|^2}}\right) &= n\varphi^\alpha(x) H_\varepsilon^{(\alpha-1)}(f(x)) H(f(x)) + \\ &+ (\alpha-1)|H_\varepsilon(f(x))|^{\alpha-2} \varphi^\alpha(x) H_\varepsilon'(f) \frac{|\nabla f|^2}{\sqrt{1+|\nabla f|^2}} + \\ &+ \alpha\varphi^{\alpha-1}(x) H_\varepsilon^{(\alpha-1)}(f(x)) \sum_{i=1}^{n} \frac{\varphi_{x_i} f_{x_i}}{\sqrt{1+|\nabla f|^2}}, \end{aligned}$$

holds, where $H_\varepsilon^{(\alpha)}(t)$ denotes $H_\varepsilon(t)|H_\varepsilon(t)|^{\alpha-1}$ and $\varphi_{x_i} = (\partial\varphi/\partial x_i)$. Taking into account (i) from (4) and the monotonicity of $H_\varepsilon(t)$, we arrive at

$$\varphi^{\alpha}|H_{\varepsilon}(f)|^{\alpha}\le\frac{\alpha\varphi^{\alpha-1}}{n}|H_{\varepsilon}(f)|\left(\sum_{i=1}^{n}\varphi_{x_i}^{2}\right)^{\frac{1}{2}}+\frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\varphi^{\alpha}H_{\varepsilon}^{(\alpha-1)}(f)\frac{f x_i}{\sqrt{1+|\nabla f|^2}}\right).$$

Integrating this expression and applying Stokes' theorem, we have

$$\int_{\Omega}\varphi^{\alpha}(x)\,|H_{\varepsilon}(f(x))|^{\alpha}dx\le\frac{\alpha}{n}\int_{\Omega}\varphi^{\alpha-1}(x)\,|H_{\varepsilon}(f(x))|^{\alpha-1}|\nabla\varphi|dx.$$

Bearing in mind that $\varphi(x)\equiv 0$ for $x\in P$, with the aid of Cauchy's inequality we arrive at

$$\int_{P}|H_{\varepsilon}(f(x))|^{\alpha}dx\le\left(\frac{\alpha}{n}\right)^{\alpha}\int_{\Omega}|\nabla\varphi(x)|^{\alpha}dx.$$

Using (ii) from (4) and passing to the limit as $\varepsilon\to 0$, we find that

$$\int_{P}|H(f(x))|^{\alpha}dx\le\left(\frac{\alpha}{n}\right)^{\alpha}\int_{\Omega}|\nabla\varphi(x)|^{\alpha}dx.$$

The proof is now concluded by passing to the greatest lower bound over all the $\varphi(x)$ admissible in (2).

The proof of Corollary 1 is based on the well-known property of n-parabolicity of Euclidean space R^n and, namely, that any compactum $\varphi(x)$ has zero n-capacitance with respect to infinity in R^n.

Let us fix a bounded open set $Q\in\mathbb{R}^n$ such that $\bar{Q}\cap P=\phi$. Let $B=B(0;R)$ be a ball with center at the origin and radius R chosen so that $B=\bar{Q}\cup P$ is fulfilled. Let us consider arbitrarily functions $\varphi_1(x)$ and $\varphi_2(x)$ that are admissible when $\alpha=n$ in (2) for capacitors $(\bar{Q},\,\partial B;\,\mathbb{R}^n\backslash\bar{B})$ and $(P,\,\bar{Q}\cup\partial B;\,\mathbb{R}^n\backslash(\bar{B}\cap\bar{Q}))$, respectively. This implies that $\varphi_1(x)(1-\varphi_2(x))$ is admissible for $(\bar{Q},\,\partial B\cup P;\,\mathbb{R}^n\backslash(\bar{B}\cap P))$. By the definition of capacitance,

$$\operatorname{cap}_n(\bar{Q},\,\partial B\cup P;\,\mathbb{R}^n\backslash(\bar{B}\cap P))\le\int_{B(0;R)}|\nabla\varphi_1(1-\varphi_2)-\varphi_1\nabla\varphi_2|^n dx\le$$
$$\le 2^{n-1}\int_{B(0;R)}|\nabla\varphi_1|^n dx+2^{n-1}\int_{B(0;R)\backslash\bar{Q}}|\nabla\varphi_2|^n dx.$$

since $\nabla\varphi_2\equiv 0$ on Q. Passing now to the greatest lower bounds over $\varphi_1(x)$ and $\varphi_2(x)$, we obtain

$$2^{1-n}\operatorname{cap}_n(\bar{Q},\,\partial B\cup P;\,\mathbb{R}^n\backslash(\bar{B}\cup P))\le$$
$$\le\operatorname{cap}_n(\bar{Q},\,\partial B;\,\mathbb{R}^n\backslash\bar{B})+\operatorname{cap}_n(P,\,\bar{Q}\cup\partial B;\,\mathbb{R}^n\backslash(\bar{B}\cup\bar{Q})).\tag{5}$$

Since $\bar{Q}\cap P=\phi$, according to the well-known (see, for example, [2, p. 61]) property of sets of zero capacitance, the last term in (5) vanishes.

To estimate the capacitance of $(\bar{Q},\,\partial B;\,\mathbb{R}^n\backslash\bar{B})$ we use standard arguments. We fix $r=\max_{x\in Q}|x|$, $R>r$ and consider $\varphi(x)$ equal to one and zero on $B(0;R)$ and $\mathbb{R}^n\backslash B(0;R)$, respectively, and having the form $\varphi(x)=(\ln(R/|x|)(\ln(R/r)^{-1}$ inside the ball fiber $B(0;R)\backslash B(0;R)$. Clearly, $\varphi(x)$ is admissible for $(\bar{Q},\,\partial B;\,\mathbb{R}^n\backslash\bar{B})$ and the desired estimate has the form

$$\operatorname{cap}_n(\bar{Q},\,\partial B;\,\mathbb{R}^n\backslash\bar{B})\le\frac{1}{\left(\ln\frac{R}{r}\right)^n}\int_{B(0;R)\backslash B(0;R)}\frac{dx}{|x|^n}=\frac{\omega_n}{\left(\ln\frac{R}{r}\right)^{n-1}},$$

where ω_n is the $(n-1)$-dimensional Lebesgue measure of the unit sphere $\partial B(0;1)$.

Now, bearing in mind (3) for $\alpha=n$, we obtain

$$\int_Q |H(f(x))|^n dx \le \frac{\omega_n}{\left(\ln\frac{R}{r}\right)^{n-1}}$$

for an arbitrarily sufficiently large R. Consequently, letting $R \to \infty$, we obtain $|H(f(x))| \equiv 0$ on Q. Since Q is taken arbitrarily from $\mathbb{R}^n \backslash P$, we have $H(f(x)) \equiv 0$ for $x \in \mathbb{R}^n \backslash P$. Assertion b) follows immediately from the strict monotonicity of $H(t)$. In the case where $H(t)$ is not necessarily strictly monotonic in a neighborhood of its zero, we obtain that $f(x)$ specifies the graph of a minimal surface over $\mathbb{R}^n \backslash P$. The validity of Assertion a) then follows from the well-known results of Bombieri et al. [4] and Simons [5].

2. The assertions presented below state that a natural characteristic of the behavior of the mean curvature of the solutions of (1) is the distance to the boundary of the domain of existence.

THEOREM 2. Let $f(x)$ be a solution of (1) in $\Omega \in \mathbb{R}^n$. Then

$$\sup_{x\in\Omega} |H(f(x))| \operatorname{dist}(x;\partial\Omega) \le 1. \tag{6}$$

Equality is attained in the case (6) where $f(x)$ describes the graph of a maximal hemisphere over $B(0; R)$.

COROLLARY 2. Let $f(x)$ be a solution of (1) in ball $B(0; R)$ of radius R with center at zero. Then $|H(f(0))| \le 1/R$.

For surfaces of constant mean curvature and mean curvature bounded away from zero this assertion was obtained by Bernstein [6] and Finn [7], respectively.

In [8] Finn investigated in detail the properties of surfaces describing the phenomenon of capillarity that are a special case in (1) when the right-hand side is the linear function $H = at + b$, where a and b are constant and $a > 0$. The assertion formulated below offers a somewhat different estimate for the behavior of such solutions than the one given in [8].

COROLLARY 3. Let $H(t):\mathbb{R} \to \mathbb{R}$ be a strictly increasing function and suppose that $f(x)$ is a solution of (1) with right-hand side $H(t)$ in $\Omega \subset \mathbb{R}^n$. Then

$$H^{-1}\left(-\frac{1}{\operatorname{dist}(x,\partial\Omega)}\right) \le f(x) \le H^{-1}\left(\frac{1}{\operatorname{dist}(x,\partial\Omega)}\right), \quad H^{-1}\circ H(t) = t.$$

It is also of interest to establish the exact value of the functional on the right-hand side of (6) for an arbitrary and, especially, noncompact Ω. Below we give a partial solution of this problem for a special form of Ω.

And, specifically, we consider two classes of domains in $\mathbb{R}^n$. For an arbitrary integer ρ, where $1 \le \rho \le n$, we denote by $\mathbb{R}^n$ a cylindrical fiber which, to within motion and homothety of $\mathbb{R}^n$, has the form of the coordinate product $\Pi_{n,\rho} = \gamma^{n-\rho} \times D\rho(a)$, where $\gamma^{n-\rho}$ is an $(n-\rho)$-dimensional plane and $D\rho(a)$ is a ρ-dimensional disk of radius $a > 0$. We denote by $D(a; b) = D^n(b) \backslash D^n(a)$ a ball fiber of width $(b-a)$ with internal radius $a > 0$.

The following is true:

THEOREM 3. Suppose that Ω coincides with some domain $\pi_{n,\rho}$. Then for any solution $f(x)$ of (1)

$$\sup_{x\in\Omega} |H(f(x))| \operatorname{dist}(x;\partial\Omega) \le \frac{\rho}{n}, \tag{7}$$

is fulfilled in $\Omega = \Pi_{n,\rho}$ and equality is attained when, for example, $f(x)$ describes a hypersurface of constant mean curvature: $f(x) = [1 - \sum_{i=1}^{\rho} x_i^2]^{1/2}$.

When Ω is the ball fiber $D(a; b)$,

$$\sup_{x\in\Omega} |H(f(x))| \operatorname{dist}(x;\partial\Omega) \le \frac{1}{2}\cdot\frac{n}{n-1} \tag{8}$$

Remark. When $n = 2$, there exist surfaces of constant mean curvature over $D(a; b)$ for which the value of the functional on the left-hand side of (8) is equal to 1/2. When $n \geq 3$, the value 1/2 is attained asymptotically for a family of surfaces of constant mean curvature over $D(a; b)$ as $a \to 0$. That is, the estimate given in (8) is, in a well-known sense, exact as $n \to \infty$.

Proof of Theorem 2. Let $z \in \Omega$ and suppose that $B(z; R)$ is a ball of maximum radius inscribed in Ω with center at z. If $H \not\equiv 0$ [trivial case for (6)], then by virtue of Corollary 1, $\mathbb{R} < +\infty$. Let us denote by $S_\lambda(r)$ an n-dimensional sphere in $\mathbb{R}^{n+1}$ with center at $(z_1;\ldots; z_n; \lambda)$ and radius r, $r < \mathbb{R}$. We consider an arbitrary solution $f(x)$ of (1) in Ω with mean curvature $H(f(x))$ and let F be the graph of this solution. Clearly, for sufficiently large numbers $\lambda > f(z)$ the intersection $F \cap S_\lambda(r)$ is empty. Let us find the greatest lower bound λ_0 of such values of λ. Since $B(z, r)$ lies compactly in $B(z, \mathbb{R})$, the sphere $S_{\lambda_0}(r)$ is tangent to surface F at some point $(x_0; f(x_0))$, where $|x_0 - z| < r$; moreover, the normal vectors of the sphere and F have opposite signs at the point of tangency. Comparing the principal curvatures of both surfaces and taking into account that $S_{\lambda_0}(r)$ lie nowhere below F, we obtain $H(f(x)) \leq H_s \equiv 1/r$. Now, using the monotonicity property of $H(t)$ and the fact that the point $(z; f(z))$ of F does not lie higher than $(x_0; f(x))$, that is, $f(x_0) \geq f(z)$, we find $H(f(z)) \leq H(f(x_0)) \leq 1/r$. Passing to the limit in this inequality as $r \to R$, we then obtain $H(f(z))\mathrm{dist}(z; d\Omega) \leq 1$.

Inequality in the other direction for values of $\lambda < f(z)$ is proved analogously, which implies the validity of (6).

The proof of Theorem 3 is based on the extended comparison principle of mean curvatures for tangent, not necessarily compact, surfaces. Let us first consider the case of a domain of the form $\Omega = n_{n,p}$, where $1 \leq \rho \leq n-1$ [the case $\rho = n$ is omitted since for it $\Pi_{n,n}$ is a ball and (7) becomes (6)]. Using the equivariant properties of the mean curvature of a hypersurface in R^{n+1} under homothetic and motion transformations, we can assume that $\pi_{n,p}$ has a form of a cylindrical fiber

$$\Pi_{n,p} = \left\{ x = (x_1; \ldots; x_n) \in R^n : \sum_{i=1}^{\rho} x_i^2 < 1 \right\}.$$

Let us arbitrarily consider $z \in \Pi_{n,p}$ and assume that $R = \mathrm{dist}(z, \partial\Pi)$. We denote by v and w the projections of vector $x \in R^{n+1}$ onto the mutually orthogonal subspaces $R^{n+1} : V = \left\{ x \in R^{n+1} : x_i = 0,\ 1 \leq i \leq \rho \right\}$ and $W = \left\{ x \in R^{n+1} : x_i = 0,\ \rho+1 \leq i \leq n+1 \right\}$ respectively. Next we consider the λ-parametric torus family

$$\mathcal{T}_\lambda(M) = \left\{ x = (w; v) \in R^{n+1} : |w|^2 + (|v - \lambda e_{n+1}| - M)^2 = r^2 \right\},$$

where r and M are fixed numbers such that $0 < r < R < M < +\infty$. From the representation of torus $\mathcal{T}_\lambda(M)$ we see that its projection along coordinate vector e_{n+1} is a compact, strictly interior subset of fiber $\Pi_{n,p}$.

Let $f(x)$ be an arbitrary solution of (1) in $\Pi_{n,p}$ and suppose that λ_0 is the greatest lower bound of λ for which $\mathcal{T}_\lambda(M)$ lies strictly above F. As above, we verify that $\lambda_0 < +\infty$ and $\mathcal{T}_{\lambda_0}(M)$ is tangent to F at some $(x_0; f(x))$, where $x_0 \in R^n$ lies strictly inside the projection of $\mathcal{T}_{\lambda 0}(M)$ along e_{n+1}. Comparing the mean curvature of F at the point of tangency to mean curvature $H_{\mathcal{T}}$ of $\mathcal{T}_{\lambda_0}(M)$, we arrive at $H(f(x_0)) \leq H_{\mathcal{T}}(x_0)$.

Suppose that x_0 has expansion $x_0 = w_0 + v_0$. Calculating the mean curvature of the torus directly, we find

$$H_{\mathcal{T}}(x_0) = \frac{1}{r}\left[\frac{p}{n} + \frac{n-p}{n}\,\frac{|v_0 - \lambda_0 e_{n+1}| - M}{|v_0 - \lambda_0 e_{n+1}|}\right] \leq \frac{1}{r}\left[\frac{p}{n} + \frac{n-p}{n}\,\frac{r}{M+r}\right].$$

Next $f(x_0) \geq f(z)$, since $(z; f(z))$ lies below any point of the torus, including also $(x_0; f(x_0))$. Consequently,

$$H(f(x_0)) \leq H(f(x_0)) \leq \frac{1}{r}\left[\frac{p}{n} + \frac{n-p}{n}\,\frac{r}{M+r}\right]$$

and after letting $M \to +\infty$, $r \nearrow R$, we obtain

$$H(f(z)) \leq \frac{p}{n}\,\frac{1}{R} = \frac{p}{n}\,\frac{1}{\mathrm{dist}(z; d\Omega)},$$

It is easy to verify that $f_1(x) = -f(x)$ is also a solution of (1) with right-hand side $H_1(t) = -H_1(-t)$ and $H_1'(t) \geq 0$. Consequently,

$$-H(f(z)) \leq \frac{p}{n} \frac{1}{\text{dist}(z; d\Omega)}$$

and the proof of the first case is complete.

To prove (8) when $\Omega = D(a, b)$, $0 < a < b$, we arbitrarily consider a $z \in D(a, b)$ and an $f(x)$ that is a solution of (1) in Ω. We consider the lower part of the horizontal torus

$$\Phi(R;\rho) = \left\{ x \in R^{n+1} : x_{n+1}^2 + \left(R - \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}\right)^2 = \rho^2 \right\},$$

which is a figure of rotation in R^{n+1} for a circle arc given by $g(t) = -(\rho^2 - (R-t)^2)^{1/2}$. The mean curvature of the torus is computed from the general formula for axially symmetric surfaces in which we let $x = (x_1, \ldots, x_n; x_{n+1})$, $t = \sqrt{\sum_{i=1}^{n} x_i^2}$,

$$H_\Phi(x) = \frac{1}{nt^{n-1}} \frac{d}{dt} \left(\frac{g'(t) t^{n-1}}{\sqrt{1+g'^2(t)}} \right) = \frac{nt - R(n-1)}{n\rho t}.$$

It is easy to estimate the mean curvature on the domain, $R - \rho < t < R + \rho$.

$$H_\Phi(x) \leq H_{\Phi,\max} = \frac{R + n\rho}{n\rho(R+\rho)}.$$

Let $\rho_0 = \text{dist}(z, \partial\Omega) - \varepsilon$, $R_0 = |z|$, where $\varepsilon > 0$ is sufficiently small. Note that $\Phi(\rho_0, R)$ is projected along e_{n+1} strictly inside $D(a, b)$ and, using the method described above, we obtain

$$H(f(z)) \leq \frac{R_0 + n\rho_0}{n\rho_0(R_0 + \rho_0)}$$

or, taking into account that $\text{dist}(z; \partial\Omega) \leq |z|$,

$$H(f(z))\,\text{dist}(z; \partial\Omega) \leq \frac{|z| + n\rho_0}{n|z| + n\rho_0} \frac{\text{dist}(z; \partial\Omega)}{\text{dist}(z \cdot \partial\Omega) - \varepsilon}.$$

Letting $\varepsilon \to 0$, we find $\rho_0 \to \text{dist}(z; \partial\Omega)$, that is,

$$H(f(z))\,\text{dist}(z; \partial\Omega) \leq \frac{|z| + n\,\text{dist}(z; \partial\Omega)}{n(|z| + \text{dist}(z; \partial\Omega))} \leq \frac{1+n}{2n}.$$

Inequality from below is proved analogously for $f_1(x) = -f(x)$.

3. Example. Let $H(t) = c(1-t^n)^{-1/n}$, where $c > 0$. Then $H'(t) \geq 0$. Let us consider a surface of rotation in R^{n+1} given by the graph of the function $x_{n+1} = g(|x|)$, $|x| = \sqrt{\sum_{i=1}^{n} x_i^2}$, with mean curvature equal to $H(|x|)$. The function $q = q(\rho)$ must satisfy

$$\frac{1}{n\rho^{n-1}} \frac{d}{d\rho} \left(\frac{g'(\rho)\rho^{n-1}}{\sqrt{1+g'^2(\rho)}} \right) = H(\rho) = c(1-\rho^n)^{-\frac{1}{n}}.$$

Solving this equation for $g(\rho)$ and letting $g'(\rho) = 0$, we get

$$\frac{g'(\rho)\rho^{n-1}}{\sqrt{1+g'^2(\rho)}} = \int_0^\rho nct^{n-1}(1-t^n)^{-\frac{1}{n}}dt = \frac{cn}{n-1}\left(1-(1-\rho^n)^{\frac{n-1}{n}}\right)$$

or

$$\frac{g'(\rho)}{\sqrt{1+g'^2(\rho)}} = \frac{cn}{n-1}\,\frac{1-(1-\rho^n)^{\frac{n-1}{n}}}{\rho^{n-1}} = \frac{cn}{n-1}\varphi(\rho).$$

We verify directly that

$$\varphi'(\rho) = \frac{n-1}{\rho^n}\,\frac{1-(1-\rho^n)^{\frac{1}{n}}}{(1-\rho^n)^{\frac{1}{n}}} > 0$$

for $0 \le \rho < 1$. Consequently, on the interval of existence, $g(\rho)$ is convex and increases monotonically for $0 \le \rho < \rho_0$, where ρ_0 is determined from $\lim_{t\to\rho_0-0} g'(t) = +\infty$ or $\varphi(\rho_0)(cn/n-1) = 1$. Taking for the domain the unit ball, that is, $\rho_0 = 1$, we find $(cn/n-1)$ or $c = (n-1/n)$. Finally, the enumerated properties of $g(t)$ and $\varphi(t)$ imply that $H(\rho) = H_1(g(\rho))$, where $H_1'(t) \ge 0$, i.e., $x_{n+1} = g(|x|)$ is a solution of (1) in a unit ball with center at zero for which

$$\lim_{|x|\to 1} H_1 g(|x|) = \lim_{\rho\to 1}\frac{n-1}{n}(1-\rho^n)^{-\frac{1}{n}} = +\infty.$$

Here it is important to note that solution $g(\rho)$ itself is necessarily bounded.

4. The possibility noted above that the behavior of the mean curvature, when approaching any point of the boundary of the domain of existence, is singular becomes impossible for isolated singularities. Namely, the following holds:

THEOREM 4. Suppose that Ω is a domain in $\mathbb{R}^n$ and $q \in \Omega$ is a fixed point. Let $f(x)$ be a solution of (1) in $\Omega\backslash\{q\}$. Then $H(f(x))$ is a function bounded in a neighborhood of q and

$$\limsup_{x\to q} |H(f(x))| \le \frac{1}{\operatorname{dist}(q;\partial\Omega)}. \tag{9}$$

Proof. Without loss of generality, we can assume that $q = 0$. Let $R = \operatorname{dist}(0; \partial\Omega)$ and let $\Omega_1 = B(0; R)\backslash\{0\}$ be a subdomain in Ω. We first construct a special hypersurface of constant curvature over domain $D(a; b)$, where $0 < a < b < R$ and $D(a; b) = B(0; b)\backslash\overline{B(0; a)}$. We consider a solution of

$$\frac{1}{n\rho^{n-1}}\frac{d}{d\rho}\left\{\frac{\rho^{n-1}g'(\rho)}{\sqrt{1+g'^2(\rho)}}\right\} = h$$

with $h = h(a; b) = \text{const} > 0$ chosen below and with boundary conditions

$$\lim_{\rho\to a+0} g'(\rho) = -\infty, \quad \lim_{\rho\to b-0} g'(\rho) = +\infty. \tag{10}$$

The graph of $g = g(\rho)$ is the surface of rotation $g(a; b)$ with mean curvature $(a; b)$ specified over $h(a; b)$. Here the boundary conditions imply that everywhere along $\partial D(a; b)$ the space tangent to $g(a; b)$ passes orthogonally to the space $\{x_{n+1} \equiv 0\}$. For $a < \rho < b$

$$\frac{g'(\rho)}{\sqrt{1+g'^2(\rho)}} = h\rho + c\rho^{1-n},$$

where the constant $c = c(a; b)$ is determined from the boundary conditions. Using (10), we find

$$\begin{cases} ah+ca^{1-n}=-1; \\ bh+cb^{1-n}=1, \end{cases}$$

whence we obtain

$$h=h(a;b)=\frac{a^{n-1}+b^{n-1}}{b^n-a^n}; \qquad c=c(a;b)=\frac{(ab)^{n-1}(a+b)}{b^n-a^n}. \tag{11}$$

Here the minimum of $g(a; b)$ is on sphere $S(\rho_0)$ of radius ρ_0, where $g'(\rho_0) = 0$ or

$$\rho_0=\rho_0(a;b)=\left[\frac{a^{n-1}b^{n-1}(a+b)}{a^{n-1}+b^{n-1}}\right]^{\frac{1}{n}}.$$

Let us fix a and b so that $0 < a < b < R$ and consider surface $g(a; b)$ determined by a, b in (11). Let $g_\lambda(a; b)$ denote $g(a; b)$ shifted by $\lambda \in R$ along the direction of e_{n+1}. Let $f(x)$ be an arbitrary solution of (1) in Ω_1 and suppose that λ_0 is the greatest lower bound of $\lambda > f(x_0)$ (for a fixed $x_0 \in \Omega_1$)) for which $g_\lambda(a; b)$ lies strictly above surface F defined by $f(x)$. Since the global minimum of $g_{\lambda_0}(a; b)$ is on $S(\rho_0)$, for $\rho_0 = \rho_0(a; b)$, repeating the arguments of Sec. 2, we obtain

$$\max_{z\in S(\rho_0(a;b))} |H(f(z))| \le h(a;b)| = \frac{a^{n-1}+b^{n-1}}{b^n-a^n}$$

It is easy to note that the right-hand side is a decreasing function as $a \to 0$. Here

$$\lim_{a\to 0} \rho_0(a;b)=\lim_{a\to 0}\left[\frac{a^{n-1}b^{n-1}(a+b)}{a^{n-1}+b^{n-1}}\right]^{\frac{1}{n}}=0.$$

and therefore in the entire punctured ball $B(0; \rho_0(a; b))\setminus\{0\}$

$$\sup_{z\in B(0;\rho_0)\setminus\{0\}} |H(f(z))| \le \frac{a^{n-1}+b^{n-1}}{b^n-a^n} \le \frac{1}{b-a},$$

that is,

$$\lim_{z\to 0} \sup |H(f(z))| \le \lim_{a\to 0}\frac{1}{b-a} = \frac{1}{b}.$$

Passing to the limit as $b \nearrow R$, we obtain the desired relation. The theorem is proved.

5. In this section we waive the requirement that $H(t)$ increase monotonically. Then the problem of classifying integral (that is, defined in the entire $\mathbb{R}^n$) solutions of an equation with $H(t)$ as the right-hand side becomes meaningful even in dimension $n = 2$. Let us give examples illustrating how large the indicated class of solutions is:

a) any function of the form $f(x) = f(x_1, \ldots, x_n) = \varphi(a_1x_1, \ldots, a_nx_n)$, where $\varphi(t)$ is a twice-differentiable function of one variable and a_i are constants;

b) any function of the form $v(x) = \varphi(\sum_{i=1}^{\rho} (x-a_i)^2)$, where $\varphi'(0) = 0$; a_i = const and ρ is an integer such that $2 \le \rho \le n$;

c) for large values of $n \geq 7$, for example, $n \geq 7$, there exist examples of minimal (that is, $H = 0$) graphs different from a) and b). Let us present the progress in this problem for the two-dimensional case. Let $\underset{x \in M}{OSC} f(x)$ be the oscillation of $f(x)$ in the set $M \in R^n$, that is, $\underset{x \in M}{OSC} f(x) = \sup_{x \in M} f(x) - \inf_{x \in M} f(x)$.

THEOREM 5. Suppose that $f(x) = f(x_1; x_2) - C^3$ is a solution of (1) in the plane $\mathbb{R}^2$ with $H(t)$ as the right-hand side. Assume that $H(t)$ does not change sign and that

$$(i)\quad |\nabla f(x)| \neq 0; \quad \underset{x \in \mathbb{R}^2}{OSC} \arg \nabla f(x) < +\infty.$$

Then $f(x)$ is a function of one variable, that is, there exist constants $\rho_1; \rho_2$ and function $\psi(t) \in C^2(\mathbb{R}^2)$ such that $f(x_1; x_2) \equiv \varphi(\rho_1 x_1 + \rho_2 x_2)$ is fulfilled.

Proof. For our purposes we extend the concept of 2-capacitance for Riemannian manifold $(F; dS^2)$ with metric $dS^2 = \sum_{i,j=1}^{2} g_{ij} dx_i dx_j$, where $g_{ij} = g_{ij}(x)$ are components of the metric tensor and $x = (x_1; x_2) \in \mathbb{R}^2$. For example, for the graph of function $f = f(x_1; x_2)$ of class $C^1(\mathbb{R}^2)$ viewed as a surface in $\mathbb{R}^3$, components $g_{ij}(x)$ have the form

$$g_{ij}(x) = \delta_{ij} + f_{x_i} f_{x_j}, \tag{12}$$

where δ_{ij} is the Kronecker delta. For two closed nonintersecting sets P, $Q \subset \mathbb{R}^2$, we let [9]

$$\operatorname{cap}_F(P;Q) = \inf \int_{\mathbb{R}^2} \sum_{i,j=1}^{2} g_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \sqrt{g}\, dx_1 dx_2, \tag{13}$$

where $g = \det \| g_{ij} \|$, $\| g^{ij} \|$ is the matrix inverse to $\| g_{ij} \|$, and the greatest lower bound is taken over all the Lipschitz functions with compact support such that $\varphi \equiv 1$ on P and $\varphi \equiv 0$ on Q. We say that $(F; dS^2)$ has parabolic type if for any compactum $P \subset \mathbb{R}^2$ there exists a sequence $D_k \subset \mathbb{R}^2$ of open sets with compact closures such that $P \subset D_k \subset D_{k+1}$, $\bigcup_{k=1}^{\infty} D_k = \mathbb{R}^2$

$$\lim_{k \to \infty} \operatorname{cap}_F(P; \mathbb{R}^2 \setminus D_k) = 0.$$

The following property of graphs in $\mathbb{R}^3$ with mean curvature of the same sign is well known [3], [10].

LEMMA. Suppose that $F = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon x_3 = f(x_1, x_2)\}$ is the graph of $f(x_1; x_2)$ with mean curvature $H(x)$ that does not change sign in $\mathbb{R}^2$. Then $(F{:}dS_F^2)$ has parabolic type as a manifold with a metric of form (12) induced from $\mathbb{R}^2$.

Let us consider any integral solution of (1) satisfying the hypotheses of the theorem. Let

$$A = \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \colon \mathbb{R}^2 \to \mathbb{R}^2; \quad \langle A; B \rangle = \sum_{i=1}^{2} A_i B_i.$$

Then after differentiating (1) with respect to x_i, $i = 1, 2$ and performing elementary transformations, we obtain

$$\begin{aligned} 0 = f_{x_1} \operatorname{div} \frac{\partial A}{\partial x_2} - f_{x_2} \operatorname{div} \frac{\partial A}{\partial x_1} &= \operatorname{div}\left(f_{x_1} \frac{\partial A}{\partial x_2} - f_{x_2} \frac{\partial A}{\partial x_1}\right) + \\ &+ \left[\left\langle \frac{\partial A}{\partial x_1}; \nabla f_{x_2} \right\rangle - \left\langle \nabla f_{x_1}; \frac{\partial A}{\partial x_2} \right\rangle \right]. \end{aligned} \tag{14}$$

Next,

$$\frac{\partial A}{\partial x_i} = \frac{\sqrt{g}\,\nabla f_{x_i} - \frac{\partial A}{\partial x_i}\sqrt{g}}{g},$$

whence we obtain that the bracketed expression in (14) vanishes. In addition,

$$f_{x_1}\frac{\partial A}{\partial x_2} - f_{x_2}\frac{\partial A}{\partial x_1} = \frac{1}{\sqrt{g}}\,(f_{x_1}\nabla f_{x_2} - f_{x_2}\nabla f_{x_1}) -$$

$$-\frac{\nabla f}{g}\left(f_{x_1}\frac{\partial\sqrt{g}}{\partial x_2} - f_{x_2}\frac{\partial\sqrt{g}}{\partial x_1}\right) = \frac{g-1}{\sqrt{g}}\,\left(\nabla\theta - \frac{\nabla f\langle\nabla f;\nabla\theta\rangle}{g}\right),$$

where $\theta(x) = \operatorname{arctg}\left[\frac{f_{x_1}}{f_{x_2}}\right]$ is a function of class $C^2(\mathbb{R}^3)$ determined for all $x \in \mathbb{R}^2$ [by virtue of (i)]. In our notation, from (14) and (15) we find

$$\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(\frac{g-1}{\sqrt{g}}\sum_{j=1}^{2} g^{ij}\frac{\partial\theta}{\partial x_j}\right)=0. \tag{16}$$

From the last relation we see that (16) is an elliptic equation of divergent type that is linear with respect to $\theta(x)$ and is related to the Laplace–Beltrami equation. Here, by virtue of (i), $\theta = \theta(x)$ is a bounded solution of (16). Let us show that $\theta(x_1, x_2) \equiv$ const. To this end we apply arguments developed in [9]. We arbitrarily fix a constant $c = \theta(q)$, where $q \in \mathbb{R}^2$, and denote by O_c the connectivity component of a set on which $\theta(x) > c$. Assume that ∂O_C is not empty and consider the function $w(x) = \theta(x) - c$ on O_c and equal to zero for $x \in \mathbb{R}^2\backslash O_C$. Let us arbitrarily fix a compactum $P \in O_C$ and consider the exhaustion of $\mathbb{R}^2$ by open sets $D_k \supset P$ for which

$$\lim_{k\to\infty} \operatorname{cap}_F(P;\mathbb{R}^2\backslash D_k)=0.$$

The existence of such an exhaustion follows from the hypothesis of Theorem 5 and the above mentioned lemma. For any Lipschitz function $\varphi(x)$ admissible in (13) for calculating the capacitance of $(P;\ \mathbb{R}^2\backslash D_k)$, we have, by virtue of (16),

$$\int_{\mathbb{R}^2}\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(\varphi^2 w(x)\frac{g-1}{\sqrt{g}}\sum_{j=1}^{2} g^{ij}\frac{\partial w}{\partial x_j}\right)dx_1dx_2=$$
$$=\int_{\mathbb{R}^2}\left(\sum_{i,j=1}^{2}\varphi^2\frac{g-1}{\sqrt{g}}g^{ij}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}+2\varphi w\frac{g-1}{\sqrt{g}}\sum_{i,j=1}^{2} g^{ij}\frac{\partial w}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\right)dx_1dx_2. \tag{17}$$

Next note that

$$\left(\sum_{i,j=1}^{2} g^{ij}\lambda_i\mu_j\right)^2 \le \left(\sum_{i,j=1}^{2} g^{ij}\lambda_i\lambda_j\right)\left(\sum_{i,j=1}^{2} g^{ij}\mu_i\mu_j\right)$$

for any positive definite matrix $\|g^{ij}\|$ and numbers λ_i, μ_j, $1 \le i, j \le 2$. Applying the corresponding Cauchy inequality to (17), we obtain

$$\int_{\mathbb{R}^2}\varphi^2\sum_{i,j=1}^{2} g^{ij}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\frac{g-1}{\sqrt{g}}\,dx_1dx_2\le$$

$$\leq 4\int_{\mathbb{R}^2} w^2 \sum_{i,j=1}^{2} g^{ij} \frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\frac{g-1}{\sqrt{g}}\,dx_1dx_2.$$

Recalling that $w(x)$ is bounded, for example, $|w(x)| \leq M$, and $\varphi(x) \equiv 1$ is fulfilled on P, we find from the last inequality

$$\int_D \sum_{i,j=1}^{2} g^{ij} \frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\frac{g-1}{\sqrt{g}}\,dx_1dx_2 \leq$$
$$\leq 4M^2\int_{O_C} \sum_{i,j=1}^{2} g^{ij} \frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\sqrt{g}\,dx_1dx_2,$$

which, after passing to the greatest lower bound over all $\varphi(x)$, yields

$$\int_P \sum_{i,j=1}^{2} g^{ij} \frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\frac{g-1}{\sqrt{g}}\,dx_1dx_2 \leq 4M^2 \operatorname{cap}_F(P;\mathbb{R}^2\setminus D_k).$$

By virtue of the property of $\{D_k\}$, the right-hand side of this inequality can be made arbitrarily small, that is,

$$\sum_{i,j=1}^{2} g^{ij} \frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\frac{|\nabla f|^2}{\sqrt{1+|\nabla f|^2}} = 0 \tag{18}$$

for $x \in P$, and since $P \subset O_c$ is chosen arbitrarily, (18) is fulfilled for all $x \in O_c$. From (i) and the positive definiteness of $\sum_{i,j=1}^{2} g^{ij}\xi_i\xi_j$, we obtain

$$|\nabla\theta(x)| \equiv 0,\ x \in O_C,$$

and, therefore, $\partial \equiv$ const on O_c. This means that there exist constants ρ_1, ρ_2 not simultaneously equal to zero for which $\rho_2 f_{x_1} - \rho_1 f_{x_2} \equiv 0$ is fulfilled everywhere in O_c. Solving this differential equation, we obtain the desired assertion.

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