Elliptic functions: Introduction course

Vladimir G. TKACHEV

Department of Mathematics, Royal Institute of Technology
Lindstedtsvägen 25, 10044 Stockholm, Sweden
email: tkatchev@math.kth.se
URL: http://www.math.kth.se/˜tkatchev

Contents

CHAPTER 1

Elliptic integrals and Jacobi's theta functions

1.1. Elliptic integrals and the AGM: real case

1.1.1. Arclength of ellipses. Consider an ellipse with major and minor arcs $2a$ and $2b$ and eccentricity $e := (a^2 - b^2)/a^2 \in [0, 1)$, e.g.,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

What is the arclength $\ell(a; b)$ of the ellipse, as a function of a and b? There are two easy observations to be made:

(1) $\ell(ra; rb) = r\ell(a; b)$, because rescaling by a factor r increases the arclength by the same factor;
(2) $\ell(a; a) = 2\pi a$, because we know the circumference of a circle.

Of course, π is transcendental so it is debatable how well we understand it!

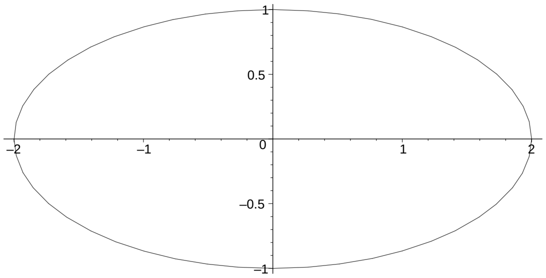


FIGURE 1. Ellipse $x^2 + \frac{y^2}{4} = 1$

The total arclength is four times the length of the piece in the first quadrant, where we have the relations

$$y = b\sqrt{1 - (x/a)^2}, \qquad y'(x) = -\frac{xb}{a^2}\frac{1}{\sqrt{1 - (x/a)^2}}.$$

Thus we obtain

$$\ell(a,b) = 4\int_0^a \sqrt{1+y'^2(x)}\,dx =$$

$$\text{substituting } z = x/a$$

$$= 4a\int_0^1 \sqrt{\frac{1-ez^2}{1-z^2}}\,dx =$$

$$= 4a\int_0^1 \frac{1-ez^2}{\sqrt{(1-ez^2)(1-z^2)}}\,dx.$$

This is an example of an *elliptic integral of the second kind.*

1.1.2. The simple pendulum. How do we compute the period of motion of a simple pendulum? Suppose the length of the pendulum is L and the gravitational constant is g. Let θ be the angle of the displacement of the pendulum from the vertical. The motion of the pendulum is governed by a differential equation

$$\theta''(t) = -\frac{g}{L}\sin\theta(t).$$

In basic calculus and physics classes, this is traditionally linearized to

$$\theta''(t) = -\frac{g}{L}\theta(t), \qquad \theta \approx 0,$$

so that the solutions take the form

$$\theta(t) = A\cos\omega t + B\sin\omega t, \qquad \omega = \sqrt{\frac{g}{L}}.$$

We obtain simple harmonic motion with frequency ω and period $2\pi/\omega$.

We shall consider the nonlinear equation, using a series of substitutions. First, note that our equation integrates to

$$\frac{1}{2}\theta'^2 - \omega^2\cos\theta = \text{const}$$

Assume that the pendulum has a maximal displacement of angle $\theta = \alpha$; then $\theta'(\alpha) = 0$ so we have

$$\frac{1}{2}\theta'^2 = \omega^2(\cos\theta - \cos\alpha),$$

and thus,

$$\theta' = \pm\omega\sqrt{2(\cos\theta - \cos\alpha)}.$$

We take positive square root before the maximal displacement is achieved. Integrating again, we obtain

$$\omega t = \int_0^\theta \frac{d\phi}{\sqrt{2(\cos\phi - \cos\alpha)}} = \frac{1}{2}\int_0^\theta \frac{d\phi}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\phi}{2}}}.$$

Substituting

$$z = \frac{\sin\frac{\phi}{2}}{\sin\frac{\alpha}{2}}, \qquad \rho = \frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}, \qquad e = \sin^2\frac{\alpha}{2} \in [0,1),$$

we obtain

$$\omega t = \int_0^\rho \frac{dz}{\sqrt{(1-z^2)(1-ez^2)}}.$$

At maximal displacement $\theta = \alpha$ we have $\rho = 1$, so the first time where maximal displacement occurs is given by

$$\frac{T}{4} = \frac{1}{\omega} \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-ez^2)}},$$

where T is the *period* of the oscillation (which is four times the time needed to achieve the maximal displacement). These are examples of *elliptic integrals of the first kind*.

Finally, we should point out that actually computing the function $\theta(t)$ involves inverting the function

$$\rho \to \int_0^\rho \frac{dz}{\sqrt{(1-z^2)(1-ez^2)}}.$$

1.1.3. The arithmetic-geometric mean iteration. The arithmetic-geometric mean of two numbers a and b is defined to be the common limit of the two sequences $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ determined by the algorithm

$$\begin{aligned} a_0 &= a, \qquad b_0 = b \\ a_{n+1} &= \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 0, 1, 2\ldots, \end{aligned} \tag{1.1}$$

where $b_n + 1$ is always the positive square root of $a_n b_n$.

Note that a_1 and b_1 are the respective arithmetic and geometric means of a and b, a_2 and b_2 the corresponding means of a_1 and b_1, etc. Thus the limit

$$M(a,b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \tag{1.2}$$

really does deserve to be called the arithmetic-geometric mean (AGM) of a and b. This algorithm first appeared in papers of Euler and Lagrange (sometime before 1785), but it was Gauss who really discovered (in the 1790s at the age of 14) the amazing depth of this subject. Unfortunately, Gauss published little on the AGM during his lifetime. [1]

THEOREM 1.1. *Let a and b be positive real numbers. Then the limits in (1.2) do exist and coincide.*

PROOF. We will assume that $a \geq b > 0$, and we let $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ be as in (1.1). The usual inequality between arithmetic and geometric means,

$$\frac{a_n + b_n}{2} \geq \sqrt{a_n b_n}$$

---

[1] By May 30th, 1799, Gauss had observed, purely computationally, that

$$\frac{1}{M(1,\sqrt{2})} \qquad \text{and} \qquad \frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^4}}$$

agreed to at least eleven (!) decimal places. He commented in his diary that this result "will surely open up a whole new field of analysis" — a claim vindicated by the subsequent directions of nineteenth-century mathematics. The inverse of the above (indefinite) integral is the lemniscate sine, a function Gauss studied in some detail. He had recognized it as a doubly periodc function by the year 1800 and hence had anticipated one of the most important developments of Abel and Jacobi: the inverse of algebraic integrals.

Figure 2. GAUSS Carl Friedrich (1777-1855)

immediately implies that $a_n \geq b_n$ for all $n \geq 0$. Actually, much more is true: we have

$$a_1 \geq a_2 \geq \ldots \geq a_n \geq a_{n+1} \geq \ldots \geq b_{n+1} \geq b_n \geq \ldots \geq b_1 \geq b_0 \tag{1.3}$$

and

$$0 \leq a_n - b_n \leq 2^{-n}(a-b). \tag{1.4}$$

To prove (1.3), note that $a_n \geq b_n$ and $a_{n+1} \geq b_{n+1}$ imply

$$a_n \geq \frac{a_n + b_n}{2} = a_{n+1} \geq b_{n+1} = \sqrt{a_n b_n} \geq b_n,$$

and (1.3) follows. From $b_{n+1} \geq b_n$ we obtain

$$a_{n+1} - b_{n+1} \leq a_{n+1} - b_n = 2^{-1}(a_n - b_n),$$

and (1.4) follows by induction. From (1.3) we see immediately that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, and (1.4) implies that the limits are equal. □

Thus, we can use (1.2) to define the arithmetic-geometric mean $M(a,b)$ of a and b. Below we list the simple properties of the AGM.

Fact 1: $M(a,a) = a$;
Fact 2: $M(a,b) = M(b,a)$;
Fact 3: $M(a,0) = 0$;
Fact 4: $M(a,b) = M(a_1,b_1) = M(a_2,b_2) = \ldots$;
Fact 5: $M(\lambda a, \lambda b) = \lambda M(a,b)$;
Fact 6: $M(a,b) = M\left(\frac{a+b}{2}, \sqrt{ab}\right)$.

In particular, the latter relation leads us to

$$M(1,x) = M\left(\frac{1+x}{2}, \sqrt{x}\right),$$

which shows that the AGM $f(x) := M(1,x)$ is a solution to the following functional equation

$$f(x) = \frac{1+x}{2}\, f\left(\frac{2\sqrt{x}}{1+x}\right).$$

Our next result shows that the AGM is not as simple as indicated by what we have done so far. We now get our first glimpse of the depth of this subject.

Theorem 1.2 (Gauss, 1799). *Let a and b are positive reals. Then*

$$\frac{1}{M(a,b)} = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}$$

Proof 1. As before, we assume that $a \geq b > 0$. Let $I(a,b)$ denote the above integral, and set $\mu = M(a,b)$. Thus we need to prove

$$I(a,b) = \frac{\pi}{2\mu}.$$

The key step is to show that

$$I(a,b) = I(a_1,b_1). \tag{1.5}$$

Let us introduce a new variable ϕ' such that

$$\sin\phi = \frac{2a\sin\phi'}{a+b+(a-b)\sin^2\phi'}. \tag{1.6}$$

Note that $0 \leq \phi' \leq \frac{\pi}{2}$ corresponds to $0 \leq \phi \leq \frac{\pi}{2}$. To see this we consider the function

$$f(t) := \frac{2at}{a+b+(a-b)t^2}.$$

Then

$$f'(t) = 2a\frac{(a+b-(a-b)t^2)}{(a+b+(a-b)t^2)^2} \geq \frac{2ab}{(a+b+(a-b)t^2)^2} > 0$$

which means that $f(t)$ increasing in $[0,1]$. On the other hand,

$$f(0) = 0, \qquad f(1) = 1,$$

which yields our claim.

Now, we note that

$$\frac{d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} = \frac{d\phi'}{\sqrt{a_1^2\cos^2\phi' + b_1^2\sin^2\phi'}}. \tag{1.7}$$

Indeed, one can find from (1.6)

$$\cos\phi = \frac{2\cos\phi'\sqrt{a_1^2\cos^2\phi' + b_1^2\sin^2\phi'}}{a+b+(a-b)\sin^2\phi'} \tag{1.8}$$

and it follows (by straightforward manipulations) that

$$\sqrt{a^2\cos^2\phi + b^2\sin^2\phi} = a\frac{a+b-(a-b)\sin^2\phi'}{a+b+(a-b)\sin^2\phi'}. \tag{1.9}$$

Then (1.7) follows from these formulas by taking the differential of (1.6).

Iterating (1.5) gives us

$$I(a,b) = I(a_1,b_1) = I(a_2,b_2) = \ldots,$$

so that

$$I(a,b) = \lim_{n\to\infty} I(a_n,b_n) = I(\mu,\mu) = \frac{\pi}{2\mu},$$

since the functions

$$\frac{1}{\sqrt{a_1^2 \cos^2 \phi' + b_1^2 \sin^2 \phi'}}$$

converge uniformly to the constant function $\frac{1}{\mu}$.

$\square$

REMARK 1.1.1. Here we prove (1.7).

$$\begin{aligned}
\cos^2 \phi &= 1 - \frac{4a^2 \sin^2 \phi'}{((a+b)+(a-b)\sin^2 \phi')^2} = \\
&= \frac{(a+b)^2 + 2(a^2-b^2)\sin^2 \phi' + (a-b)^2 \sin^4 \phi' - 4a^2 \sin^2 \phi'}{((a+b)+(a-b)\sin^2 \phi')^2} = \\
&= \text{(using our notation for } a_1 \text{ and } b_1\text{)} = \\
&= \frac{4a_1^2 - 4(2a_1^2 - b_1^2)\sin^2 \phi' + 4(a_1^2 - b_1^2)\sin^2 \phi'}{((a+b)+(a-b)\sin^2 \phi')^2} = \\
&= \frac{4(a_1^2 \cos^4 \phi' + 4b_1^2 \sin^2 \phi' \cos^2 \phi'}{((a+b)+(a-b)\sin^2 \phi')^2} =
\end{aligned}$$

and (1.8) follows.

To prove (1.9) we note that

$$\begin{aligned}
a^2 \cos^2 \phi + b^2 \sin^2 \phi &= \frac{4a^2 \cos^2 \phi'(a_1^2 \cos^2 \phi' + b_1^2 \sin^2 \phi') + 4a^2 b^2 \sin^2 \phi'}{((a+b)+(a-b)\sin^2 \phi')^2} = \\
&= 4a^2 \frac{a_1^2(1-\sin^2 \phi')^2 + b_1^2 \sin^2 \phi'(1-\sin^2 \phi') + b^2 \sin^2 \phi'}{((a+b)+(a-b)\sin^2 \phi')^2} = \\
&= \text{(using the old variables } a \text{ and } b\text{)} = \\
&= a^2 \frac{(a+b)^2(1-\sin^2 \phi')^2 + 4ab \sin^2 \phi'(1-\sin^2 \phi') + 4b^2 \sin^2 \phi'}{((a+b)+(a-b)\sin^2 \phi')^2} = \\
&= a^2 \frac{(a+b)^2 - 2(a-b)(a+b)\sin^2 \phi' + (a-b)^2 \sin^4 \phi'}{((a+b)+(a-b)\sin^2 \phi')^2} =
\end{aligned}$$

which implies (1.9).

Finally, (1.6) gives

$$\cos \phi \, d\phi = 2a \frac{a+b-(a-b)\sin^2 \phi'}{(a+b+(a-b)\sin^2 \phi')^2} \cos \phi' \, d\phi'.$$

We have for the left-hand side from (1.8)

$$\cos \phi \, d\phi = 2 \cos \phi' \frac{\sqrt{a_1^2 \cos^2 \phi' + b_1^2 \sin^2 \phi'}}{a+b+(a-b)\sin^2 \phi'} \, d\phi$$

which yields

$$\begin{aligned}
\sqrt{a_1^2 \cos^2 \phi' + b_1^2 \sin^2 \phi'} \, d\phi &= a \frac{a+b-(a-b)\sin^2 \phi'}{a+b+(a-b)\sin^2 \phi'} \, d\phi' = \\
&= \sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi} \, d\phi'
\end{aligned}$$

and (1.7) is proven.

REMARK 1.1.2. Another proof is due to Carlson [**3**]. It uses the representation

$$\frac{2}{\pi}\int_0^{\pi/2}\frac{d\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}} \tag{1.10}$$

with the further substitution

$$u:=\frac{1}{2}\left(t-\frac{ab}{t}\right).$$

Exercise 1.1.1. Consider the harmonic-geometric mean iteration

$$\alpha_{n+1}=\frac{2\alpha_n\beta_n}{\alpha_n+\beta_n},\qquad \beta_{n+1}=\sqrt{\alpha_n\beta_n}.$$

Show, for $\alpha_0,\beta_0\in(0,\infty)$, that the above iteration converges to

$$H(\alpha_0,\beta_0)=\frac{1}{M(1/\alpha_0,1/\beta_0)}.$$

Exercise 1.1.2. Prove (1.10) and the recurrence relation

$$\int_{-\infty}^{+\infty}\frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}}=\int_{-\infty}^{+\infty}\frac{dt}{\sqrt{(a_1^2+t^2)(b_1^2+t^2)}}.$$

1.2. Lemniscates and elastic curves

> . . . Today, the elastic curve has been largely forgotten, and he lemniscate has suffered the worse fate of being relegated to the polar coordinates section of calculus books. There it sits next to the formula for arc length in polar coordinates, which can never be applied to the lemniscate since such texts know nothing of elliptic integrals. . .
> D.A. Cox, [**6**].

1.2.1. Arclength of lemniscate. A *lemniscate*[2] was discovered by Jacob BERNOULLI in 1694.

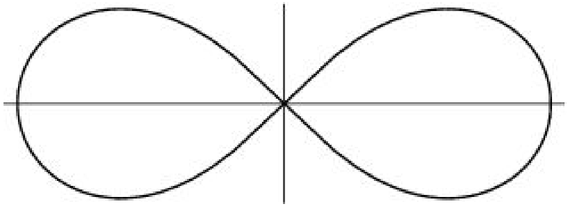


FIGURE 3. The lemniscate

He gives the equation in the form

$$(x^2+y^2)^2=2a^2(x^2-y^2) \tag{1.11}$$

---

[2] Animation and formulas: http://www.mathcurve.com/courbes2d/lemniscate/lemniscate.shtml
Formulas and more calculations: http://mathworld.wolfram.com/Lemniscate.html
Applet: http://iaks-www.ira.uka.de/home/egner/linkages/lemnis.html

and explains that the curve has "the form of a figure 8 on its side, as of a band folded into a knot, or of a lemniscus, or of a knot of a French ribbon" [3].

FIGURE 4. Jacob Bernoulli (1654 – 1705)

A polar curve also called *Lemniscate of Bernoulli* which is the locus of points the product of whose distances from two points (called the *foci*) is a constant. Letting the foci be located at $(\pm a, 0)$, the Cartesian equation is

$$[(x-a)^2+y^2][(x+a)^2+y^2]=a^4.$$

The polar coordinates are given by

$$r^2 = 2a^2\cos(2\theta). \tag{1.12}$$

Let now fix $a = 1/\sqrt{2}$ so that (1.12) can be written as $r^2(\theta)=\cos 2\theta$. Using the formula for arc length in polar coordinates, see that the total arc length L is

$$L = 4\int_0^{\pi/4}(r^2+r'^2)^{1/2}\,d\theta = 4\int_0^{\pi/4}\frac{d\theta}{\sqrt{\cos 2\theta}}.$$

The substitution $\cos 2\theta = \cos^2\phi$ transforms this to the integral

$$L = 4\int_0^{\pi/2}\frac{d\phi}{\sqrt{1+\cos^2\phi}} = 4\int_0^{\pi/2}\frac{d\phi}{\sqrt{2\cos^2\phi+\sin^2\phi}} = \frac{2\pi}{M(\sqrt{2},1)},$$

which links the Gauss AGM $M(\sqrt{2},1)$ and the arc length of the lemniscate.

Finally, letting $t=\cos\phi$ we obtain

$$L = 4\int_0^{\pi/2}\frac{d\phi}{\sqrt{2\cos^2\phi+\sin^2\phi}} = 4\int_0^1\frac{dt}{\sqrt{1-t^4}}\ . \tag{1.13}$$

1.2.2. Elastic Curves. More interesting is that the integral in the right hand side of (1.13) had been discovered by Jacob Bernoulli three years earlier in 1691.

This was when Bernoulli worked out the equation of the so-called *elastic curve.* The situation is as follows: a thin elastic rod is bent until the two ends are perpendicular to a given line H. After introducing cartesian coordinates as indicated on Figure 5 and letting

---

[3] In 1694 Jacob Bernoulli published a curve in *Acta Eruditorum.* Following the protocol of his day, he gave this curve the Latin name of *lemniscus*, which translates as a pendant ribbon to be fastened to a victor's garland. He was unaware that his curve was a special case of the Ovals of Cassini. His investigations on the length of the arc laid the foundation for later work on elliptic functions.

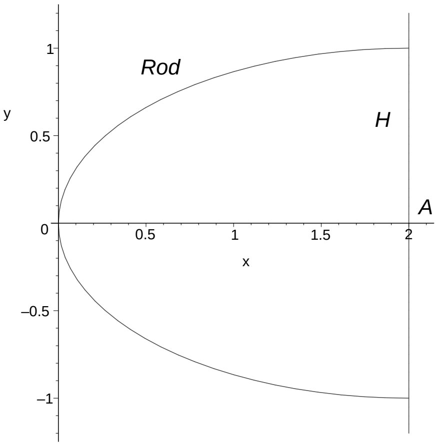


FIGURE 5. Elastic curve

a denote $0A$, Bernoulli was able to show that the upper half of the curve is given by the equation

$$y = \int_0^x \frac{t^2 dt}{\sqrt{a^4 - t^4}}, \tag{1.14}$$

where $0 \leq x \leq a$.

It is convenient to assume that $a = 1$. But as soon as this is done, we no longer know *how long the rod is.* In fact, (1.14) implies that the arc length from the origin to a point (x, y) on the rescaled elastic curve is

$$\ell(x) = \int_0^x (1 - t^4)^{-1/2} dt. \tag{1.15}$$

Thus the half-length of the whole rod is

$$\ell := \int_0^1 \frac{dt}{\sqrt{1 - t^4}}.$$

How did Bernoulli get from here to the lemniscate? He was well aware of the transcendental nature of the elastic curve, and so he used a standard seventeenth century trick to make things more manageable: he sought an algebraic curve whose rectification should agree with the rectification of the elastic curve.

Since Bernoulli's solution involved the arc length of the elastic curve, it was natural for him to seek an algebraic curve with the same arc length. Very shortly thereafter, he found the equation of the lemniscate. So the arc length of the lemniscate was known well before the curve itself.

1.2.3. Euler's identity. Throughout the 18th century the elastic curves and the lemniscate appeared in many papers. A lot of work was done on the integrals (1.14) and (1.15). A notable work on the elastic curve was Euler's paper of 1786. Namely, Euler gives approximations to the above integrals and, more importantly, proves the following amazing result

THEOREM 1.3 (EULER'S IDENTITY).

$$\int_0^1 \frac{dt}{\sqrt{1-t^4}} \cdot \int_0^1 \frac{t^2 dt}{\sqrt{1-t^4}} = \frac{\pi}{4}. \tag{1.16}$$

The proof is given in Exercise 1.2.2. We prove a more general assertion following [**10**]

THEOREM 1.4 (The generalized elastic curves, [**10**]). *Let*

$$f_n(x) := \int_0^x \frac{t^n dt}{\sqrt{1-t^{2n}}}$$

be the generalized elastic curve. Let us denote by $R_n = f_n(1)$ *the so called main radius, and by* L_n *the length of the curve from* $x = 0$ *to* $x = 1$*. Then*

$$L_n R_n = \frac{\pi}{2n}. \tag{1.17}$$

REMARK 1.2.1. One can easily observe that for $n = 1$ (1.17) is the well-known identity since

$$L_1 = \frac{\pi}{2}, \qquad R_1 = 1.$$

PROOF. We have

$$R_n = \int_0^1 \frac{t^n dt}{\sqrt{1-t^{2n}}},$$

and one can easily find that

$$L_n = \int_0^1 \frac{dt}{\sqrt{1-t^{2n}}}.$$

Integrate the relation

$$d(t^k\sqrt{1-t^{2n}}) = \frac{kt^{k-1}t^{k-1} - (k+n)t^{2n+k-1}}{\sqrt{1-t^{2n}}}\, dt$$

from 0 to 1 to produce the recursive formula

$$\int_0^1 \frac{t^{k-1}dt}{\sqrt{1-t^{2n}}} = \frac{k+n}{k}\int_0^1 \frac{t^{2n+k-1}dt}{\sqrt{1-t^{2n}}}. \tag{1.18}$$

The value $k = n+1$ in (1.18) yields

$$R_n = \frac{2n+1}{n+1}\int_0^1 \frac{t^{3n}dt}{\sqrt{1-t^{2n}}}.$$

Then the value $k = 3n+1$ produces

$$R_n = \frac{4n+1}{3n+1}\int_0^1 \frac{t^{5n}dt}{\sqrt{1-t^{2n}}},$$

so we have

$$R_n = \frac{2n+1}{n+1} \times \frac{4n+1}{3n+1}\int_0^1 \frac{t^{5n}dt}{\sqrt{1-t^{2n}}}.$$

Iterating we obtain, after m steps,

$$R_n = \prod_{j=1}^{m} \frac{2jn+1}{(2j-1)n+1} \times \int_0^1 \frac{t^{(2m+1)n}dt}{\sqrt{1-t^{2n}}}. \tag{1.19}$$

The next step is to justify the passage to the limit in (1.19) as $m \to \infty$, with n fixed. Observe that the left hand side is independent of m, so it remains R_n after $m \to \infty$. The difficulty in passing to the limit is that the product in (1.19) diverges. The general term p_j satisfies

$$1 - p_j = -\frac{n}{(2j-1)n+1}$$

and the divergence of the product follows from that of the harmonic series. The divergence is cured by introducing scaling factors both in the integral and the product.

PROPOSITION 1.1. *The functions*

$$\frac{1}{\sqrt{2m+1}} \prod_{j=1}^{m} \frac{2jn+1}{(2j-1)n+1} \qquad \textit{and} \qquad \sqrt{2m+1} \int_0^1 \frac{t^{(2m+1)n}dt}{\sqrt{1-t^{2n}}}$$

have non-zero limits as $m \to \infty$.

Therefore from (1.19) we obtain

$$R_n = \lim_{m\to\infty} \prod_{j=1}^{2m} (jn+1)^{(-1)^j} \times \int_0^1 \frac{t^{(2m+1)n}dt}{\sqrt{1-t^{2n}}},$$

where we have employed

$$\prod_{j=1}^{2m} (jn+1)^{(-1)^j} = \prod_{j=1}^{m} \frac{2jn+1}{(2j-1)n+1}$$

in order to simplify the notation. A similar argument shows that

$$\begin{aligned} L_n &= \prod_{j=1}^{m} \frac{(2j-1)n+1}{2(j-1)n+1} \times \int_0^1 \frac{t^{2mn}dt}{\sqrt{1-t^{2n}}} \\ &= \lim_{m\to\infty} \prod_{j=1}^{2m} (jn+1)^{(-1)^{j+1}} \times \int_0^1 \frac{t^{2mn}dt}{\sqrt{1-t^{2n}}} \end{aligned}$$

The final step is to introduce the auxiliary quantities

$$A_n := \int_0^1 \frac{t^{n-1}dt}{\sqrt{1-t^{2n}}} \qquad \text{and} \qquad B_n := \int_0^1 \frac{t^{2n-1}dt}{\sqrt{1-t^{2n}}}.$$

We now show that the quotient L_n/A_n can be evaluated explicitly and that the value of A_n is elementary. This produces an expression for L_n. A similar statement holds for R_n/B_n and B_n.

Observe first that (after change of the variable)

$$A_n = \int_0^1 \frac{t^{n-1}dt}{\sqrt{1-t^{2n}}} = \frac{1}{n}\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2n}, \tag{1.20}$$

and similarly $B_n = 1/n$. Now consider the recursion (1.18) for odd multiples of n to

$$A_n = \lim_{m\to\infty} \prod_{j=1}^{2m} (jn)^{(-1)^j} \times \int_0^1 \frac{t^{(2m+1)n-1}dt}{\sqrt{1-t^{2n}}}$$

and similarly the even multiples of n yield

$$B_n = \frac{1}{n} \lim_{m\to\infty} \prod_{j=1}^{2m+1} (jn)^{(-1)^{j+1}} \times \int_0^1 \frac{t^{(2m+1)n-1}dt}{\sqrt{1-t^{2n}}}$$

in the exact manner as the derivation of (1.19). Therefore using the last identities, and passing to the limit as $m \to \infty$ so that the integrals disappear, we obtain

$$\frac{L_n}{A_n} = \prod_{j=1}^{\infty} \left[(jn+1)^{(-1)^{j+1}} \times (jn)^{(-1)^{j+1}}\right]$$

so (1.20) yields

$$L_n = \frac{\pi}{2n} \times \prod_{j=1}^{\infty} \left[(jn+1)^{(-1)^{j+1}} \times (jn)^{(-1)^{j+1}}\right].$$

Similarly, using $B_n = 1/n$,

$$R_n = \times \prod_{j=1}^{\infty} \left[(jn+1)^{(-1)^{j}} \times (jn)^{(-1)^{j}}\right].$$

The formula $Rn \times Ln = \pi/2n$ follows directly from here. □

Exercise 1.2.1. Prove Proposition 1.1.

Exercise 1.2.2. Give another proof of Theorem 1.4 by using the Β and Γ Euler's functions:

$$\mathrm{B}(\alpha,\beta) = \int\limits_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.21}$$

and the fact that $\Gamma(1/2) = \sqrt{\pi}$.

1.2.4. Addendum: The lemniscate and its "relatives".

1.2.4.1. *Circle and lemniscate.* There are several methods for drawing a lemniscate. The easiest is illustrated below. Draw a circle and then extend a diameter to become a secant. The center of the lemniscate O will be $\sqrt{2}$ times the radius of the circle. Through O draw several segments cutting the circle. The pattern of the lemniscate emerges in the first quadrant (see Figure 6).

Exercise 1.2.3. Prove the mentioned property for the unit circle. **Hint**: use the secant line equation (with fixed angle α)

$$(x,y) = (-\sqrt{2} + t\cos\alpha, t\sin\alpha), \qquad \alpha \in \left[\frac{\pi}{4}, \frac{\pi}{4}\right],$$

where the interval $t \in [\tau_1(\alpha), \tau_2(\alpha)]$ is defined by substitution in the circle equation $x^2+y^2 = 1$. Then the required equation of the lemniscate is given by

$$(X,Y) = (\tau_1(\alpha) - \tau_2(\alpha)) \times (\cos\alpha, \sin\alpha).$$

It's also worth noting that the lemniscate is the inverse (in the sense of inversive geometry) of the hyperbola relative to the circle of radius $k = a\sqrt{2}$ where a is defined by (1.11). In other words, if we draw a line emanating from the origin and it strikes the lemniscate at the radius s, then it strikes the hyperbola at the radius R where $sR = k^2$.

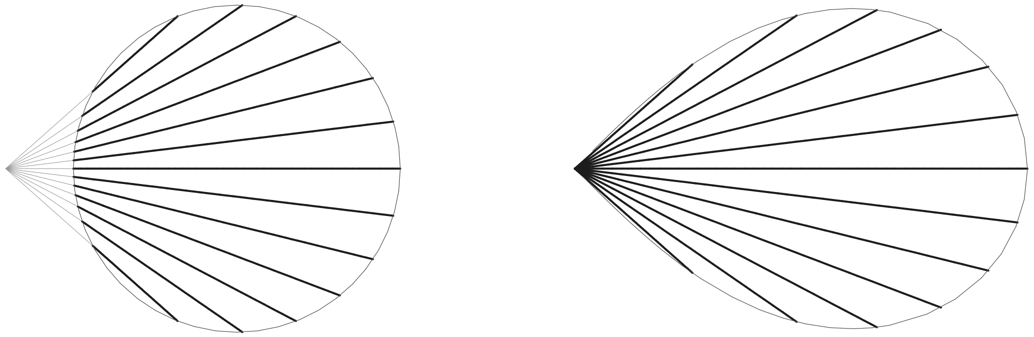
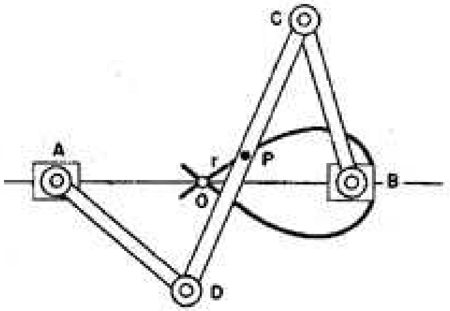


Figure 6. The lemniscate and the circle

1.2.4.2. *A lemniscate "machine".* Another method is based on the mechanical interpretation of the main lemniscate property and is illustrated by Figure 7.



AB = CD = a $\sqrt{2}$.

AD = BC = a.

P and O are midpoints of
DC and AB, resp.

$r^2 = a^2 \cos 2\theta$.

Figure 7. lemniscate "machine" [**9**]

Exercise 1.2.4. Give an "explanation" of the lemniscate machine.

1.2.4.3. *Cassinian Ovals.* Jacob Bernoulli was not aware that the curve he was describing was a special case of Cassini Ovals which had been described by Cassini in 1680.

Cassinian oval describe a family of curves. It is defined as the locus of points P such that the *product* of distances $|PF_1||PF_2| = b^2$ is constant. Here F_1 and F_2 are two fixed points (foci) and b is a constant. It is analogous to the definition of ellipse, where *sum* of two distances is replace by product. Let the distance between the foci be $2a$. Then a special case is the lemniscate of Bernoulli when $a = b$.

Exercise 1.2.5. Prove that the polar representation of the Cassinian Ovals is given by $r^4 + a^4 - 2r^2a^2 \cos 2\theta = b^4$.

Cassinian ovals are the intersection of a torus and a plane in certain position. Let a be the inner radius of a torus whose generating circle has radius R (see Figure 8). Cassinian oval *is the intersection of a plane parallel to the torus' axis and* R *distant from it.* If $a = 2R$, then it is the lemniscate of Bernoulli. Note that these tori in the figure are not identical. (Obs!: Arbitrary slice of a torus are not Cassinian ovals).

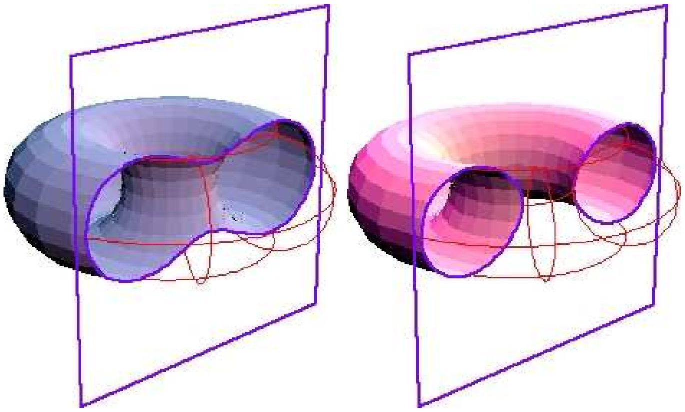


Figure 8. Cassinian ovals as intersection of a torus and a plane

Exercise 1.2.6. Prove the preceding assertion. **Hint**: Use the Cartesian representation of the torus

$$(\sqrt{x^2 + y^2} - a)^2 + z^2 = R^2.$$

1.3. Euler's addition theorem

1.3.1. Fagnano's Theorem on the lemniscate. Unlike the elastic curve, the story of the lemniscate in the 18th century is well known, primarily because of the key role it played in the development of the theory of elliptic integrals. One early worker was Giulio Carlo Fagnano (1682–1766). He, following some ideas of Johann Bernoulli, Jacob's younger brother, studied the ways in which arcs of ellipses and hyperbolas can be related.

One result, known as Fagnano's Theorem, states that the sum of two appropriately chosen arcs of an ellipse can be computed algebraically in terms of the coordinates of the points involved[4]. He also worked on the lemniscate, starting with the problem of halving

---

[4] These researches of Fagnano's were published in the period 1714–1720 in an obscure Venetian journal and were not widely known. In 1750 he had his work republished, and he sent a copy to the Berlin Academy. It was given to Euler for review on December 23, 1751. Less than five weeks later, on January 27, 1752, Euler read a paper giving new derivations for Fagnano's results on elliptic and hyperbolic arcs. By 1753 he had a general addition theorem for lemniscatic integrals, and by 1758 he had the addition theorem for elliptic integrals.

that portion of the arc length of the lemniscate which lies in one quadrant. Subsequently he found methods for dividing this arc length into n equal pieces, where $n = 2^m$, $3 \cdot 2^m$ or $5 \cdot 2^m$.

We formulate the simplest case of Fagnano's Theorem — the duplication of the lemniscate arc length.

Theorem 1.5 (Fagnano's Doubling Theorem). *Let* $0 < u < \sqrt{\sqrt{2}-1}$ *and*

$$r = \frac{2u\sqrt{1-u^4}}{1+u^4}.$$

Then

$$\int_0^r \frac{dt}{\sqrt{1-t^4}} = 2\int_0^u \frac{dt}{\sqrt{1-t^4}}.$$

Proof. First we note that the function

$$r = f(u) := \frac{2u\sqrt{1-u^4}}{1+u^4}$$

have as its derivative

$$f'(u) = 2\,\frac{(u^4+2\,u^2-1)\,(u^4-2\,u^2-1)}{(1+u^4)^2\,\sqrt{1-u^4}},$$

so it increasing in $[0, u_0]$ with u_0 being the least positive root of $f'(u) = 0$. Clearly, $u_0 = \sqrt{\sqrt{2}-1}$.

On the other hand

$$1 - f(u)^4 = 1 - \frac{16u^4(1-u^4)^2}{(1+u^4)^4} = \frac{(u^4+2\,u^2-1)^2\,(u^4-2\,u^2-1)^2}{(1+u^4)^4}.$$

It follows that

$$\frac{df}{\sqrt{1-f^4(u)}} = 2\frac{du}{\sqrt{1-u^4}}$$

and the result follows. □

1.3.2. Addition theorems. The simplest example of a function which has an *algebraic* addition theorem is the exponential function

$$\phi(u) = e^u.$$

It follows that

$$e^u \cdot e^v = e^{u+v},$$

or

$$\phi(u) \cdot \phi(v) = \phi(u+v).$$

Such an equation offers a means of determining the value of the function for the sum of two quantities as arguments, when the values of the function for the two arguments taken singly are known.

It is called an *addition theorem.*

In the example just cited the relation among $\phi(u)$, $\phi(v)$ and $\phi(u+v)$ is expressed through an algebraic equation, and consequently the addition theorem is called *algebraic addition theorem.*

FIGURE 9. LEONARD EULER (1707-1783)

The sine function has the algebraic addition theorem

$$\begin{aligned}\sin(u+v) &= \sin u\cos v + \cos u \sin v = \\ &= \sin u\sqrt{1-\sin^2 v} + \sin v\sqrt{1-\sin^2 u}.\end{aligned} \tag{1.22}$$

We also have

$$\tan(u+v) = \frac{\tan u + \tan v}{1-\tan u \tan v}$$

Another result is the previous Fagnano's duplication theorem, which can be reformulated as follows: let $\phi(u)$ be defined as the solution to

$$u = \int_0^{\phi(u)} \frac{dt}{\sqrt{1-t^4}}.$$

Then

$$\phi(2u) = \phi(u+u) = \frac{2\phi(u)\sqrt{1-\phi(u)^4}}{1+\phi(u)^4}.$$

THEOREM 1.6 (EULER'S ADDITION THEOREM). *Let*

$$f(x) := (1-x^2)(1-k^2x^2).$$

Then

$$\int_0^x \frac{dt}{\sqrt{f(t)}} + \int_0^y \frac{dt}{\sqrt{f(t)}} = \int_0^z \frac{dt}{\sqrt{f(t)}}, \tag{1.23}$$

where

$$z = \frac{x\sqrt{f(y)} + y\sqrt{f(x)}}{1-k^2x^2y^2}. \tag{1.24}$$

PROOF. We follow a method of proving the Euler theorem due to Darboux [**1**, p. 73]. Let us consider the equation

$$\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} = 0. \tag{1.25}$$

Obviously, that (1.25) defines a level set of the function $z = z(x, y)$ defined by (1.23). If we set

$$u = \int_0^x \frac{dt}{\sqrt{f(t)}},$$

$$v = \int_0^y \frac{dt}{\sqrt{f(t)}},$$

then the integral identity (1.23) can be represented in the form

$$u + v = A,$$

where A is a suitable constant.

On the other hand, equation (1.25) can be replaced by the system

$$\left\{ \begin{array}{l} \frac{dx}{dt} = \sqrt{(1-x^2)(1-k^2x^2)} \\ \frac{dy}{dt} = -\sqrt{(1-y^2)(1-k^2y^2)}. \end{array} \right. \tag{1.26}$$

Squaring (1.26), we get

$$\left\{ \begin{array}{l} \left(\frac{dx}{dt}\right)^2 = (1-x^2)(1-k^2x^2) \\ \left(\frac{dy}{dt}\right)^2 = (1-y^2)(1-k^2y^2). \end{array} \right. \tag{1.27}$$

Let us now differentiate these equations:

$$\frac{d^2x}{dt^2} = x(2k^2x^2 - 1 - k^2), \qquad \frac{d^2y}{dt^2} = y(2k^2y^2 - 1 - k^2).$$

This implies

$$y\frac{d^2x}{dt^2} - x\frac{d^2y}{dt^2} = 2k^2xy(x^2 - y^2),$$

or

$$\frac{d}{dt}\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right) = 2k^2xy(x^2 - y^2). \tag{1.28}$$

On the other hand, it follows from (1.27) that

$$y^2\left(\frac{dx}{dt}\right)^2 - x^2\left(\frac{dy}{dt}\right)^2 = (y^2 - x^2)(1 - k^2x^2y^2). \tag{1.29}$$

Dividing (1.28) by (1.29), we get

$$\frac{\frac{d}{dt}\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right)}{y\frac{dx}{dt} - x\frac{dy}{dt}} = \frac{2k^2xy\left(y\frac{dx}{dt} + x\frac{dy}{dt}\right)}{k^2x^2y^2 - 1},$$

or

$$\frac{d}{dt}\ln\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right) = \frac{d}{dt}\ln(k^2x^2y^2 - 1).$$

Thus, we have

$$y\frac{dx}{dt} - x\frac{dy}{dt} = C(k^2x^2y^2 - 1).$$

Taking (1.26) into account, we get

$$\frac{x\sqrt{(1-y^2)(1-k^2y^2)}+y\sqrt{(1-x^2)(1-k^2x^2)}}{1-k^2x^2y^2}=C.$$

This is the desired algebraic form of the integral of the equation (1.25). The theorem is proved. □

COROLLARY 1.1. *Let $k=0$. Then the assertion of the theorem is equivalent to (1.22).*

1.3.3. Jacobi's functions: preliminaries. The last considerations lead us to the most popular Jacobian elliptic functions which are

- *sine amplitude elliptic function* — $\operatorname{sn}(x,k)$,
- *cosine amplitude elliptic function* — $\operatorname{cn}(x,k)$,
- *delta amplitude elliptic function* — $\operatorname{dn}(x,k)$.

These functions may be defined via the inverse of the incomplete elliptic integrals as follows:

$$x=\int\limits_0^{\operatorname{sn}(x,k)}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$x=\int\limits_1^{\operatorname{cn}(x,k)}\frac{dt}{\sqrt{(1-t^2)(k'^2+k^2t^2)}}$$

$$x=\int\limits_1^{\operatorname{dn}(x,k)}\frac{dt}{\sqrt{(1-t^2)(t^2-k'^2)}}.$$

The second argument of the functions k — is a *modulus* of the elliptic function and

$$k':=\sqrt{1-k^2}$$

is a *complimentary modulus.* The eight remaining Jacobian elliptic functions can be conveniently defined via the general identity relations

$$\operatorname{fg}(x,k)=\frac{\operatorname{fe}(k,x)}{\operatorname{ge}(k,x)}\qquad\qquad \text{e,f,g=s,c,d,n,}$$

where $\operatorname{ff}(x,k)$ is interpreted as unity.

In this section we examine the simplest property only. To treat Jacobi's function in more detail we need to extend them into complex plane which will be given in the further sections.

PROPOSITION 1.2. *The following identities hold*

$$\operatorname{sn}^2(x,k)+\operatorname{cn}^2(x,k)=1, \tag{1.30}$$

$$\operatorname{dn}^2(x,k)+k^2\operatorname{sn}^2(x,k)=1. \tag{1.31}$$

Proof. Let $0 \leq x \leq 1$ be fixed, and $u := \mathrm{sn}(x,k)$, $v := \mathrm{cn}(x,k)$. Then we have by the definition

$$x = \int_1^v \frac{dt}{\sqrt{(1-t^2)(k'^2+k^2t^2)}} = \left| \begin{array}{ccc} s & = & \sqrt{1-t^2} \\ dt & = & -\frac{sds}{\sqrt{1-s^2}} \end{array} \right| =$$

$$= \int_0^{\sqrt{1-v^2}} \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}},$$

which clearly implies $u = \sqrt{1-v^2}$. Thus, (1.30) is proven. The second identity is proved in a similar way.

□

Exercise 1.3.1. Show that

$$\begin{array}{rclrcl} \mathrm{sn}(x,0) & = & \sin x, & \mathrm{sn}(x,1) & = & \tanh x, \\ \mathrm{cn}(x,0) & = & \cos x, & \mathrm{cn}(x,1) & = & \frac{1}{\cosh x}, \\ \mathrm{dn}(x,0) & = & 1, & \mathrm{dn}(x,1) & = & \frac{1}{\cosh x}. \end{array}$$

Now, we have an important consequence of Theorem 1.6

Corollary 1.2. *Addition/substraction formulae for sine, cosine and delta amplitude Jacobian functions are*

$$\mathrm{sn}(x \pm y;k) = \frac{\mathrm{sn}(x;k)\,\mathrm{cn}(y;k)\,\mathrm{dn}(y;k) \pm \mathrm{cn}(x;k)\,\mathrm{dn}(x;k)\,\mathrm{sn}(y;k)}{1 - k^2\,\mathrm{sn}^2(x;k)\,\mathrm{sn}^2(y;k)}$$

$$\mathrm{cn}(x \pm y;k) = \frac{\mathrm{cn}(x;k)\,\mathrm{cn}(y;k) \pm \mathrm{sn}(x;k)\,\mathrm{dn}(x;k)\,\mathrm{sn}(y;k)\,\mathrm{dn}(y;k)}{1 - k^2\,\mathrm{sn}^2(x;k)\,\mathrm{sn}^2(y;k)}$$

$$\mathrm{dn}(x \pm y;k) = \frac{\mathrm{dn}(x;k)\,\mathrm{dn}(y;k) \mp k^2\,\mathrm{sn}(x;k)\,\mathrm{cn}(x;k)\,\mathrm{sn}(y;k)\,\mathrm{cn}(y;k)}{1 - k^2\,\mathrm{sn}^2(x;k)\,\mathrm{sn}^2(y;k)}$$

The following property provides an easy application of the Addition Theorem.

Proposition 1.3. *Let*

$$K = K(k) := \int_0^1 \frac{dz}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

be the complete integral of the first kind. Then the following identities hold

$$\mathrm{sn}(K,k) = 1, \qquad \mathrm{sn}(\frac{K}{2},k) = \frac{1}{1+\sqrt{k^2-1}}. \tag{1.32}$$

Clearly, that $K(k)$ plays the role of $\frac{\pi}{2}$ for the Jacobi sine function. Hint for the proof: write

$$\mathrm{sn}(\frac{K}{2},k) = \mathrm{sn}(K - \frac{K}{2},k).$$

1.4. Theta functions: preliminaries

1.4.1. Theta functions as solutions of the Heat Conduction Problem. Theta functions appear appear in Bernoulli's *Ars Conjectandi* [1713] and in the number-theoretic investigations of Euler [1773] and Gauss [1801], but come into full flower only in Jacobi's *Fundamenta Nova* [1829].

We shall introduce the theta functions by considering a specific heat conduction problem. Namely, in this way the theta functions occur in J. Fourier's *La Théorie Analytique de la Chaleur* [1822].

Let θ be the temperature at the time t at any point in a solid material whose conduction properties are uniform and isotropic. Then, if ρ is the material's density, s is its specific heat, and k its thermal conductivity, θ satisfies the partial differential equation

$$\varkappa \Delta \theta = \frac{\partial \theta}{\partial t}, \tag{1.33}$$

where $\varkappa = s/\rho$ is termed the diffusivity and Δ is the Laplace operator. In the special case where there is no variation of temperature in the x- and y-directions of a rectangular Cartesian frame $Oxyz$, the heat flow is everywhere parallel to the z-axis and the heat conduction reduces to the form

$$\varkappa \frac{\partial^2 \theta}{\partial z^2} = \frac{\partial \theta}{\partial t}, \tag{1.34}$$

and $\theta = \theta(z, t)$.

The specific problem we shall study (in an ideal form) is the flow of heat in an infinite slab of material, bounded by the planes $z = 0$, $z = \pi$, when the conditions over each boundary plane are kept uniform at every time t. The heat flow is then entirely in the z-direction and equation (1.34) is applicable.

First, suppose the boundary conditions are that the faces of the slab are maintained at zero temperature, i.e. $\theta = 0$ for $z = 0, \pi$ and all t. Initially, at $t = 0$ suppose

$$\theta(z, 0) = f(z), \qquad 0 < z < \pi.$$

Then the method of separation of variables leads to the solution

$$\theta(z, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 \varkappa t} \sin nz, \tag{1.35}$$

where b_n are Fourier coefficients determined by the equation

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(z) \sin nz \, dz. \tag{1.36}$$

Exercise 1.4.1. Show the validity of (1.35). **Hint**: consider the Fourier expansion of $f(z)$ in the sin-series and show that $e^{-m^2 \varkappa t} \sin mz$ solves (1.34) with $f_m(z) = \sin mz$.

In the special case where

$$f(z) = \pi \delta(z - \frac{1}{2}\pi)$$

(with $\delta(z)$ to be the Dirac's unit impulse function), the slab is initially at zero temperature everywhere, except in the neighborhood of the midplane $z = \frac{\pi}{2}$, where the temperature is very high. To achieve this high temperature, it will be necessary to inject a quantity of heat h (joules) per unit area into this plane to raise its temperature from zero h is given by

$$h = \rho s \pi \int_{\pi/2-0}^{\pi/2+0} \delta(z - \frac{1}{2}\pi)\, dz = \rho s \pi. \tag{1.37}$$

We now calculate that

$$b_n = 2\int_0^{\pi} \delta(z - \frac{1}{2}\pi) \sin nz\, dz = 2 \sin \frac{n\pi}{2}.$$

Thus, heat diffusion over the slab is governed by the equation

$$\theta(z,t) = 2\sum_{n=0}^{\infty} (-1)^n e^{-(2n+1)^2 \varkappa t} \sin(2n+1)z. \tag{1.38}$$

Writing

$$q := e^{-4\varkappa t}$$

the solution (1.38) assumes the form

$$\theta = \theta_1(z,q) = 2\sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)z. \tag{1.39}$$

Definition 1.4.1. The function $\theta_1(z,q)$ given by (1.39) is the *first* theta function of Jacobi.

1.4.2. Convergence property. The main technical result is as follows

PROPOSITION 1.4. *The first theta function $\theta_1(z,q)$ is defined by the series (1.39) for all complex values z and q such that $|q| < 1$. Moreover, the series converges uniformly in any strip $-Y \leq \operatorname{Im} z \leq Y$, where $Y > 0$.*

PROOF. Replacing the sine function by its Euler representation by exponentials we obtain the Nth partial sum of the series in (1.39)

$$\begin{aligned} S_N &:= 2\sum_{n=0}^{N} (-1)^n q^{(n+1/2)^2} \sin(2n+1)z \\ &= \frac{q^{1/4}}{i} \sum_{n=0}^{N} (-1)^n q^{n^2+n} (e^{(2n+1)zi} - e^{-(2n+1)zi}) = \\ &= \frac{q^{1/4}}{i} \sum_{n=-N}^{N} (-1)^n q^{n^2+n} e^{(2n+1)zi}. \end{aligned}$$

To establish convergence, let u_n denote the nth term of the latter series. Then for $n > 0$ we have

$$\frac{|u_{n+1}|}{|u_n|} = |q^{2n+2} e^{2zi}| = |q|^{2n+2} e^{-2y}, \quad \text{where} \quad z = x + iy. \tag{1.40}$$

As $n \to +\infty$, since $|q| < 1$, this ratio tends to zero and, by D'Alembert's test, therefore, the series converges at $+\infty$. The similar argument shows that the series converges at $-\infty$ and the assertion follows.

Now, let us suppose that z is in the strip $-Y \leq \operatorname{Im} z \leq Y$, $Y > 0$. Then yields again, by D'Alembert's test and majorant principle that S_N converges uniformly. $\square$

COROLLARY 1.3. *$\theta_1(z, q)$ is an integral holomorphic function of z.*

COROLLARY 1.4. *$\theta_1(z, q)$ is a 2π-periodic function of z.*

REMARK 1.4.1. An alternative notation (Gauss' form) is to write

$$q = e^{i\pi\tau}, \tag{1.41}$$

where now the imaginary part of τ must be positive to give $|q| < 1$:

$$\operatorname{Re} \tau > 0. \tag{1.42}$$

In this notation we have

$$i\theta_1(z|q) = \sum_{n=-\infty}^{\infty} (-1)^n e^{(n+1/2)^2 \pi i \tau + (2n+1)iz}.$$

1.4.3. Four Theta Functions. Incrementing z by a quarter period, we define the *second* theta function θ_2 thus:

$$\begin{aligned} \theta_2(z, q) := \theta_1(z + \frac{\pi}{2}, q) = \\ &= 2\sum_{n=0}^{+\infty} q^{(n+1/2)^2} \cos(2n+1)z \\ &= \sum_{n=-\infty}^{+\infty} q^{(n+1/2)^2} e^{(2n+1)zi}. \end{aligned}$$

Evidently, that θ_2 is an *even* integral function of z with period 2π. The next pair is

$$\theta_3(z, q) = \sum_{n=-\infty}^{+\infty} q^{n^2} e^{2inz}, \tag{1.43}$$

and

$$\theta_4(z, q) = \theta_3(z - \frac{\pi}{2}, q) = \sum_{n=-\infty}^{+\infty} (-1)^n q^{n^2} e^{2inz}.$$

The latter two functions are periodic (of z) with period π which follows from their Fourier expansions

$$\begin{aligned} \theta_3(z, q) &= 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nz, \\ \theta_4(z, q) &= 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz. \end{aligned} \tag{1.44}$$

1.4.4. Theta functions and the AGM. Now we return to the central Gauss' observation in his AGM treatises: the AGM solution in terms of theta functions. To do this we restrict ourselves by the reduced theta series. Namely, we suppose that

$$z = 0,$$

so, our previous formulae provide the following definition.

The basic functions are defined for $|q| < 1$ by

$$\begin{aligned} P(\tau) &:= \theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} = 2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \dots, \\ Q(\tau) &:= \theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2q^1 + 2q^4 + 2q^9 + 2q^{16} + \dots, \\ R(\tau) &:= \theta_4(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = 1 - 2q^1 + 2q^4 - 2q^9 + 2q^{16} - \dots, \end{aligned}$$

with $\theta_j(0) = 0$ and τ as in (1.41).

Theorem 1.7 (The AGM representation via theta functions).

$$\begin{aligned} \frac{\theta_3^2(q) + \theta_4^2(q)}{2} &= \theta_3^2(q^2), \\ \sqrt{\theta_3^2(q)\theta_4^2(q)} &= \theta_4^2(q^2). \end{aligned} \tag{1.45}$$

Proof. First we observe that

$$\theta_4(q) = \theta_3(-q),$$

which yields

$$\theta_3(q) + \theta_4(q) = 2 \sum_{n \text{ even}} q^{n^2} = 2\theta_3(q^4). \tag{1.46}$$

Also

$$\theta_3^2(q) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} q^{n^2+m^2} = \sum_{n=0}^{\infty} r_2(n) q^n, \tag{1.47}$$

where $r_2(n)$ counts the number of ways of writing

$$n = j^2 + k^2. \tag{1.48}$$

Here we distinguish sign and permutation [so that, for example, $r_2(5) = 8$ since $5 = (\pm 2)^2 + (\pm 1)^2 = (\pm 1)^2 + (\pm 2)^2$] and set $r_2(0) := 1$. Similarly we have

$$\theta_4^2(q) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^{n+m} q^{n^2+m^2} = \sum_{n=0}^{\infty} (-1)^n r_2(n) q^n, \tag{1.49}$$

since $n^2 + m^2 \equiv n + m \mod 2$.

Now we claim that

$$r_2(2n) = r_2(n). \tag{1.50}$$

To prove this identity we fix a number $n \geq 1$ (the case $n = 0$ is trivial) and note that

$$2(a^2 + b^2) = (a - b)^2 + (a + b)^2.$$

Then evidently a pair (a, b) solves (1.48) for n if and only if $(A, B) := (a - b, a + b)$ does (1.48) for $2n$. Clearly, that the correspondence

$$(a, b) \to (A, B) = (a, b) \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

is bijective which proves our claim.

It follows from (1.47) that

$$\theta_3^2(q) + \theta_4^2(q) = 2 \sum_{n=0}^{\infty} r_2(2n) q^{2n} = 2\theta_3^2(q^2). \tag{1.51}$$

Also, (1.46) and (1.51) allow us to solve for $\theta_3(q)\theta_4(q)$:

$$\begin{aligned} \theta_3(q)\theta_4(q) &= \frac{1}{2}(\theta_3(q) + \theta_4(q))^2 - \frac{1}{2}(\theta_3^2(q) + \theta_4^2(q)) = \\ &= 2\theta_3^2(q^4) - \theta_3^2(q^2) = \\ &= \text{again (1.51)} = \\ &= \theta_4^2(q^2). \end{aligned}$$

The theorem follows. ☐

The main identity (1.45) bears an obvious resemblance with the AGM. Namely, we have

Corollary 1.5. *Let* $|q| < 1$. *Then*

$$M(\theta_3^2(q), \theta_4^2(q)) = M(\theta_3^2(q^2), \theta_4^2(q^2)) = \ldots = M(\theta_3^2(q^{2^n}), \theta_4^2(q^{2^n})) = \ldots \tag{1.52}$$

Since $\theta_3(0) = \theta_3(0) = 1$ we easily arrive at

Corollary 1.6. *Let* $|q| < 1$. *Then*

$$M(\theta_3^2(q), \theta_4^2(q)) = 1. \tag{1.53}$$

Another important property is

Corollary 1.7 (Jacobi's identity).

$$\theta_3^4(q) = \theta_4^4(q) + \theta_2^4(q). \tag{1.54}$$

Proof. We have

$$\begin{aligned} \theta_3^2(q) - \theta_3^2(q^2) &= \sum_{n=0}^{\infty} r_2(n) q^n - \sum_{n=0}^{\infty} r_2(2n) q^{2n} = \\ &= \text{by (1.50)} = \sum_{n=0}^{\infty} r_2(2n+1) q^{2n+1}. \end{aligned}$$

On the other hand,

$$\sum_{n=0}^{\infty} r_2(2n+1)q^{2n+1} = 2 \sum_{\substack{k,m=-\infty \\ k+m \text{odd}}} q^{m^2+k^2} =$$

$$= (\text{setting} \quad k = i-j, \ m = i+j+1) =$$

$$= \sum_{i,j=-\infty}^{\infty} (q^2)^{(i+1/2)^2+(j+1/2)^2} = \theta_2^2(q^2).$$

Hence

$$\theta_3^2(q^2) + \theta_2^2(q^2) = \theta_3^2(q),$$

which with the first identity in (1.45) produce

$$\theta_3^2(q^2) - \theta_2^2(q^2) = \theta_4^2(q).$$

Now (1.53) follows from the last two identities and the second identity in (1.45). □

Let us define

$$k := k(q) := \frac{\theta_2^2(q)}{\theta_3^2(q)}.$$

Then (1.53) shows that

$$k' := \sqrt{1-k^2} = \frac{\theta_4^2(q)}{\theta_3^2(q)}.$$

THEOREM 1.8 (THE AGM REPRESENTATION VIA THETA FUNCTIONS, II). *Let* $0 < k < 1$ *is given. The AGM satisfies*

$$M(1,k') = \theta_3^{-2}(q), \qquad \text{for} \quad k' = \frac{\theta_4^2(q)}{\theta_3^2(q)}, \tag{1.55}$$

where q *is the unique solution in* $(0,1)$ *to* $k = \theta_2^2(q)/\theta_3^2(q)$ *and* $k^2 + k'^2 = 1$.

CHAPTER 2

General theory of doubly periodic functions

2.1. Preliminaries

2.1.1. Holomorphic functions. Here we summarize the well-known facts about the analytic functions we need in the sequel.

We distinguish the finite complex plane $\mathbb{C}$ and its compactification $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, i.e. the Riemann sphere. We use $z = x + iy$ to indicate a complex number which plays the role of a complex variable in what follows. As usually we set $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$ for the real and imaginary parts of z respectively.

The *conjugate* to z number is denoted by $\bar{z} = x - iy$, and the *modulus* is defined as the following positive square root $|z| = \sqrt{z\bar{z}}$. The *argument* of $z \neq 0$ is a multivalued function $\arg z$ which is defined by

$$z = |z|e^{i \arg z}.$$

A function $f(z)$ of one complex variable z is called an *analytic* (or *holomorphic*) function in an open set $D \subset \mathbb{C}$ if it admits the following expansion in converging power series

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots,$$

where the disk $|z - z_0| < r$ is contained in D. In this case we have for the *radius of convergence*:

$$0 < r \leq R(z_0) := [\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}]^{-1}.$$

The Taylor coefficients a_n are found by

$$a_n = \frac{f^{(n)}(z_0)}{n!},$$

and thus

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \qquad |z - z_0| < R(z_0).$$

A function $f(z)$ is said to be analytic at z_0 is it is analytic a small neighborhood of z_0.

The point $z_0 \in D$ is called a *zero* of $f(z)$ of order $N \geq 1$ if

$$f(z) = a_N(z - z_0)^N + a_{N+1}(z - z_0)^{N+1} + a_{N+2}(z - z_0)^{N+2} + \dots, \qquad a_N \neq 0.$$

An equivalent condition is that $f(z) = (z - z_0)^N g(z)$ where $g(z)$ is an analytic function at z_0 and $g(z_0) \neq 0$.

Cauchy Integral Theorem. *Let $f(z)$ be an analytic function in D and D' is a proper subdomain, i.e. $\overline{D'} \subset D$, with rectifiable boundary Γ. Then*

$$\int_\Gamma f(z)dz = 0.$$

In particular, if $z_0 \in D'$ then

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)dz}{z - z_0}.$$

The Uniqueness Theorem. *If $f(z)$ and $g(z)$ are analytic in $D \subset \mathbb{C}$ and $f(z_k) = g(z_k)$ for some sequence $\{z_k\}_{k=1}^\infty \subset D$ which has an accumulation point in D, then $f(z) \equiv g(z)$ in D.*

The Maximum Principle. *If $f(z)$ is analytic in $D \subset \mathbb{C}$ and continuous in the closure $\overline{D}$, then for any subdomain $U \subset D$ one holds*

$$\max_{z \in U} |f(z)| = \max_{z \in \partial U} |f(z)|,$$

where ∂U denotes the boundary of U.

A holomorphic function $f(z)$ is said to be *entire* (or *integer*) if it is analytic in the whole complex plane $D = \mathbb{C}$. In other words, the entire functions are the largest class of functions holomorphic in the finite plane which are the limit functions of convergent sequences of polynomials, the convergence being uniform on every compact set.

Cauchy-Liouville Theorem. *Let $f(z)$ be a bounded entire function. Then $f(z) = const.$*

2.1.2. Singular points. Let D be an open set. A point $z_0 \in D$ of finite complex plane is said to be an *isolated singular* point of an analytic function $f(z)$ if $f(z)$ is analytic in a small punctured disk $\{z : 0 < |z - z_0| < \varepsilon\} \subset D$ and is unbounded there (otherwise, the point z_0 is called *regular* and in that case $f(z)$ can be continued up to an analytic function at z_0).

Let z_0 be an isolated singular point. Then $f(z)$ can be represented by the *Laurent series*:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \tag{2.1}$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)dz}{(z - z_0)^{n+1}}, \tag{2.2}$$

and C_ρ is the circle centered at z_0 of radius $\rho < \varepsilon$.

The following alternative is possible:

(i) if $\exists \lim_{z \to z_0} f(z) = \infty$ then z_0 is called a *pole* of $f(z)$; in this case $a_n = 0$ for sufficiently large negative $n < 0$. In other words, there is a positive integer $N \in \mathbb{N}$ such that $(z - z_0)^N f(z)$ is analytic in D. The smallest such an N is called the *order* of the pole z_0. Another equivalent definition is that z_0 is zero of $1/f(z)$ of order N.

(ii) if the limit $\lim_{z \to z_0} f(z)$ does not exist then z_0 is called an *essential* singular point of $f(z)$.

A function $f(z)$ is said to be *meromorphic* if it is holomorphic save for poles in the finite complex plane. Typical examples of meromorphic functions is rational functions or $f(z) = \tan z$.

One of the main characteristics of the holomorphic function at an isolated singularity a_0 is the constant a_1 in the Laurent series (2.1). This coefficient is called the *residue* of $f(z)$ about a point z_0. The residue of a function f around a point z_0 is also defined by

$$\operatorname*{res}_{z=z_0} f(z) = \frac{1}{2\pi i}\int_\gamma f(z)dz,$$

where γ is counterclockwise simple closed contour, small enough to avoid any other poles of f. Cauchy integral theorem implies that unless z_0 is a pole of f, its residue is zero.

Residue Theorem. *If the contour γ encloses multiple poles a in domain D, then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{a\in D} \operatorname*{res}_{z=a} f(z).$$

All the functions considered above are assumed to be single-valued, but sometimes we will consider the inverse functions, which are, normally, infinitely many valued. This leads us to new types of singularities, such as algebraic and logarithmic branch points. We refer an interested reader to monograph [**7**].

2.2. Periods of analytic functions

2.2.1. Basic properties. In all of what follows, unless otherwise stated, we will assume a function to be single-valued analytic function whose singularities do not have limit points at the finite complex plane. If $f(z)$ is such a function and if at each regular point z

$$f(z+\Omega) = f(z),$$

where Ω is a constant, then the number Ω is called a *period* of f. Zero is a *trivial* period. A function $f(z)$ having nontrivial periods is said to be *periodic*. We denote by $\mathcal{T}(f)$ the set of all periods of f.

PROPOSITION 2.1. *If $\Omega_1,\dots,\Omega_n$ are periods of a function f, then for any integers $m_1,\dots,m_n$ the number*

$$m_1\Omega_1 + \ldots + m_n\Omega_n$$

is also a period of f. In other words, $\mathcal{T}(f)$ is a module over $\mathbb{Z}$.

The proof is an easy corollary of the definition.

PROPOSITION 2.2. *Let $f(z)$ and $g(z)$ have a period Ω. Then the following functions also have the same period*

$$f(z+C), \quad f(z)\pm g(z), \quad f(z)g(z), \quad \frac{f(z)}{g(z)}, \quad f'(z).$$

PROOF. We prove the last assertion. With this goal we take the function

$$\frac{f(z+h)-f(z)}{h}, \quad h\neq 0,$$

which in view of the preceding assertions has period Ω. Therefore, at each regular point z

$$\frac{f(z+\Omega+h)-f(z+\Omega)}{h} = \frac{f(z+h)-f(z)}{h}.$$

It now remains to pass to the limit as $h\to 0$. □

PROPOSITION 2.3. *Let* $f(z) \not\equiv const$ *be a periodic function. Then there exists a* $\mu > 0$ *such that every nontrivial period of* f *satisfies the inequality*

$$|\Omega| \geq \mu > 0.$$

PROOF. Assuming the contrary, we take nontrivial periods $\{\Omega_k\}_{k=1}^{\infty}$ of f such that

$$\lim_{k\to\infty} \Omega_k = 0.$$

Since

$$\frac{f(z+\Omega_k) - f(z)}{\Omega_k} = 0$$

for any regular point z of f, it follows that

$$f'(z) = \lim_{k\to\infty} \frac{f(z+\Omega_k) - f(z)}{\Omega_k} = 0.$$

Thus $f'(z) \equiv 0$ which implies that f is a constant. $\square$

EXAMPLE 2.2.1. The simplest example of a function with period Ω is $e^{2\pi iz/\Omega}$. Clearly,

$$\mathcal{T}(e^{2\pi iz/\Omega}) = \Omega\mathbb{Z}.$$

Thus in this case there exists a primitive period, namely Ω which generates $\mathcal{T}(e^{2\pi iz/\Omega})$. Every other period is an integer multiple of the period Ω. Therefore, this function can be called a *simply periodic* function.

FIGURE 1. KARL GUSTAV JACOBI (1804-1851)

The question arises as to whether there exists a function with $n > 1$ primitive periods. Here n periods are said to be *primitive* if

- every period is a linear combination of these periods with integer coefficients
- and if not every period can be represented as such a combination of fewer fixed periods.

The answers to the question will be discussed below.

2.2.2. The Jacobi Theorem.

THEOREM 2.1. *There does not exist a nonconstant function with $n \geq 3$ primitive periods. If f is a nonconstant function and Ω, Ω' are two primitive periods of f then*

$$\operatorname{Im}\frac{\Omega'}{\Omega} \neq 0.$$

PROOF. The periods of a given function f will be represented as points in the complex plane. Then in any finite part of the plane there are only finitely many of these point-periods, because otherwise they would have a finite limit point, so that there would be a sequence $\{\Omega_k\}_1^\infty$ of periods with a finite limit, and then f would have an infinitesimal period $\Omega_m - \Omega_k$ $(m, k \to \infty)$, which is impossible, since the function is assumed to be nonconstant.

Take a nontrivial period Ω and consider the periods $m\Omega$, $m \in \mathbb{Z}$; they lie on the some straight line L. Two cases are conceivable a priori:

1) all the periods of f lie on L;

2) not all the periods of f lie on L.

Let us analyze the first case. Since the segment of L from $-\Omega$ to $+\Omega$ contains only finitely many points-periods, there is a nontrivial period with smallest modulus, and we can assume without loss of generality that Ω is precisely this period. Since all the periods lie on L, every period can be represented in the form $t\Omega$, where t is real; furthermore, t satisfies the inequality $|t| \geq 1$, since Ω is a nontrivial period with smallest modulus. We prove that t runs through only integer values. This will imply that Ω is a primitive period, and f is a simply periodic function.

Let $t = m + r$, where m is an integer and $0 \leq r < 1$. Since not only $t\Omega$ but also $m\Omega$ is a period of f, it follows that $r\Omega = t\Omega - m\Omega$ is also a period, and this, as we established, is impossible if $0 < r < 1$. Consequently, $r = 0$, i.e., $t \in \mathbb{Z}$ is an integer.

We proceed to the second case. Suppose that not all the periods of f lie on L. Denote by Ω' one of the periods not on L, and consider the triangle with vertices 0, Ω, and Ω'. By what was proved, only finitely many points can lie inside and on the boundary of this triangle. Taking instead of one of the nonzero vertices of our triangle some point-period lying inside (or on a side), we get an analogous triangle containing fewer point-periods.

Continuing this reduction, we arrive at a triangle with no point-periods inside or on its sides, except for the vertices. Without loss of generality we can assume that this "empty" triangle is the original triangle with vertices 0, Ω and Ω'. We now construct the parallelogram with vertices

$$0, \quad \Omega, \quad \Omega + \Omega' \quad \text{and} \quad \Omega'. \tag{2.3}$$

The empty triangle with vertices 0, Ω and Ω' considered earlier represents the "left" half of this parallelogram. We assert that the "right" half of the parallelogram also is an empty triangle, i.e., does not contain point-periods, neither inside nor on its sides (other than the vertices).

Indeed, if the right half contained a point-period $\widetilde{\Omega}_1$, then the left half would contain the point-period

$$\Omega + \Omega' - \widetilde{\Omega}_1 = \widetilde{\Omega}_2.$$

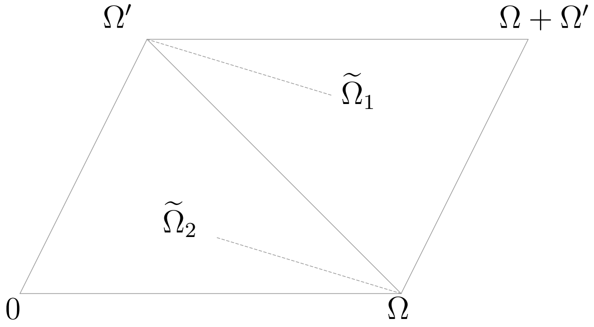


FIGURE 2.

But the left half is empty by construction. Accordingly, the parallelogram is empty. We now take some period Ω^* of our function. It has a unique representation in the form

$$\Omega^* = t\Omega + t'\Omega',$$

where t, t' are real numbers. This representation is equivalent to decomposing the vector Ω^* with respect to the vectors Ω and Ω'.

If we prove that t and t' are integers, then it will be proved that in the second case of our alternative the number of primitive periods is equal to two, and their ratio is not real. The proof of Jacobi's theorem will thereby be complete.

Thus, let $t = m + r$ and $t' = m' + r'$, where $m, m' \in \mathbb{Z}$ are integers and $0 \leq r, r' < 1$. We must prove that $r = r' = 0$.

Since $m\Omega$ and $m'\Omega'$ are periods of f,

$$\Omega_1^* = \Omega^* - m\Omega - m'\Omega' = r\Omega + r'\Omega'$$

is also a period. The point-period Ω_1^* lies in the parallelogram with vertices (2.3), and hence it must coincide with one of the vertices, since the parallelogram is empty. Thus, each of the numbers r and r' must equal 0 or 1. Since $0 \leq r, r' < 1$, it follows that $r = 0$ and $r' = 0$, as was required. □

2.3. Existence of doubly periodic functions

2.3.1. Theta functions: revisited. As the basic function we take

$$\Theta_3(z) = \theta_3(\pi z, q) = \theta_3(\pi z|\tau) = \sum_{m=-\infty}^{\infty} e^{(m^2\tau + 2mz)\pi i}$$

which agrees (1.41) and (1.43). It follows from (1.44) that

$$\Theta_3(z) = 1 + 2q\cos 2\pi z + 2q^4 \cos 4\pi z + 2q^9 \cos 6\pi z + \dots$$

is an entire function of z with period 1. On the other hand,

$$\Theta_3(z+\tau) = \sum_{m=-\infty}^{\infty} e^{(m^2\tau+2mz+2m\tau)\pi i} =$$

$$= e^{-\pi i(\tau+2z)} \sum_{m=-\infty}^{\infty} e^{[(m+1)^2\tau+2(m+1)z]\pi i} =$$

$$= e^{-\pi i(\tau+2z)} \sum_{n=-\infty}^{\infty} e^{[n^2\tau+2nz]\pi i}.$$

We see consequently, that

$$\Theta_3(z+\tau) = e^{-\pi i(\tau+2z)}\Theta_3(z). \tag{2.4}$$

Let us take the logarithm of both sides of (2.4) and take the second derivative with respect to z of both sides. We get

$$\frac{d^2}{dv^2}\ln\Theta_3(z+\tau) = \frac{d^2}{dv^2}\ln\Theta_3(z).$$

Since, moreover,

$$\frac{d^2}{dv^2}\ln\Theta_3(z+1) = \frac{d^2}{dv^2}\ln\Theta_3(z),$$

it follows that

$$\phi(z) := \frac{d^2}{dv^2}\ln\Theta_3(z)$$

is an example of a function with periods 1 and τ, the ratio of which is not real;

this function is doubly periodic.

We remark that $\varphi(z)$ is a meromorphic function, and all its poles have multiplicity *two*. Indeed, $\Theta_3(z)$ is an entire function; hence the only singularities of its logarithmic derivative are simple poles, which coincide with the zeroes of Θ_3, and thus the only singularities of $\varphi(z)$ are poles of order 2.

In addition we introduce three more theta functions:

$$\Theta_k(z) \equiv \Theta_k(z|\tau), \qquad (k=0,1,2),$$

where

$$\begin{aligned} \Theta_1(z) &= ie^{-\pi i(z-\tau/4)}\,\Theta_3(z+(1-\tau)/2);\\ \Theta_2(z) &= \ e^{-\pi i(z-\tau/4)}\,\Theta_3(z-\tau/2);\\ \Theta_4(z) &= \Theta_3(z+1/2). \end{aligned} \tag{2.5}$$

Exercise 2.3.1. Prove that $\Theta_k(z) = \theta_k(\pi z)$.

Exercise 2.3.2. Prove the following identities

$$\begin{array}{ll} \Theta_1(z\pm 1) = -\Theta_1(z); & \Theta_1(z\pm\frac{1}{2}) = \pm\Theta_2(z);\\ \Theta_2(z\pm 1) = -\Theta_2(z); & \Theta_2(z\pm\frac{1}{2}) = \mp\Theta_1(z);\\ \Theta_3(z\pm 1) = \Theta_3(z); & \Theta_3(z\pm\frac{1}{2}) = \Theta_4(z);\\ \Theta_4(z\pm 1) = \Theta_4(z); & \Theta_4(z\pm\frac{1}{2}) = \Theta_3(z). \end{array} \tag{2.6}$$

Exercise 2.3.3. Prove that all theta functions satisfy the differential equation

$$\frac{\partial^2\Theta}{\partial z^2} = 4\pi i\frac{\partial\Theta}{\partial z}.$$

Concluding this section, we note that the ratios

$$\varphi_k(z) := \frac{\Theta_k(z)}{\Theta_4(z)}, \qquad k = 1, 2, 3.$$

It follows from (2.6) that $\varphi_1(z)$ has periods 2 and τ, $\varphi_2(z)$ has periods 2 and $1+\tau$, and, finally, $\varphi_3(z)$ has periods 2 and 2τ. Each $\varphi_k(z)$ is a meromorphic function. Thus, we have a second proof of existence of doubly periodic meromorphic functions.

Definition 2.3.1. Doubly periodic *meromorphic* functions bear the name ELLIPTIC FUNCTIONS.

2.4. Liouville's theorems

2.4.1. Fundamental parallelogram. We consider elliptic functions with primitive periods Ω and Ω', and we agree to assume that the ratio

$$\tau := \frac{\Omega'}{\Omega} \in \mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

has positive imaginary part unless otherwise stated.

FIGURE 3. JOSEPH LIOUVILLE (1809-1882)

In the complex plane we take a point c and construct the parallelogram with vertices

$$c, \quad c+\Omega, \quad c+\Omega', \quad c+\Omega+\Omega'.$$

Then this passage from vertex to vertex corresponds to a circuit of the boundary of the parallelogram in the positive direction, since $\mathrm{Im}\,\frac{\Omega'}{\Omega} > 0$. Of the four vertices we include in the parallelogram only the vertex c, and of four sides we include only the ones meeting at c.

The resulting point set Π is called a *period parallelogram.* Another equivalent definition is

$$\Pi = \{z \in \mathbb{C} : z = c + r\Omega + r'\Omega', \text{ where } \ 0 \le r, r' < 1\}.$$

We say that two points z' and z'' are *congruent modulo the periods* Ω *and* Ω', or *equivalent*, if

$$z'' - z' = m\Omega + m'\Omega', \qquad m, m' \in \mathbb{Z},$$

and in this case we write

$$z' \equiv z'' \mod (\Omega, \Omega').$$

Proposition 2.4. *A period parallelogram* Π *does not contain a pair of equivalent points. On the other hand, for any point* z *there is a point in a period parallelogram equivalent to it, and, of course, it is unique.*

Proof. The first property is a consequence of the definition. To prove the second, we suppose that $z \in \mathbb{C}$ is an arbitrary point. Then there are real numbers t and t' such that

$$z - c = t\Omega + t'\Omega'.$$

Setting

$$t = m + r, \qquad t' = m' + r',$$

where m, m' are integers and

$$0 \leq r, r' < 1,$$

we find that

$$z - (c + r\Omega + r'\Omega') = m\Omega + m'\Omega'.$$

Consequently, z is equivalent to the point $c + r\Omega + r'\Omega'$, which belongs to the period parallelogram. □

Remark 2.4.1. The following simple observation is important for study elliptic functions: *We can confine ourselves to any period parallelogram.*

Due to this arbitrariness we can construct a period parallelogram in such a way that the function does not take some predetermined values on its sides (for example, does not become infinite). Such a choice of the period parallelogram is possible because an *elliptic function*, like every meromorphic function, *takes each of its values only finitely many times in a finite region.*

All points mutually congruent modulo the periods form (as is commonly said) a *regular system* or *network* of points on the plane. Corresponding to each such system is a network of parallelograms that fit together to cover the whole plane.

2.4.2. Residues theorem. Let $f(z)$ be an elliptic function with primitive periods Ω and Ω', and let the period parallelogram be chosen so that $f(z)$ is regular on its sides.

Theorem 2.2. *The sum of residues of* $f(z)$ *with respect to all the poles inside a periods parallelogram is equal to zero.*

Proof. We integrate $f(z)$ along the contour of the parallelogram Π. By Cauchy's theorem, the result of the integration is the sum of the residues of $f(z)$ with respect to all poles inside the parallelogram Π, multiplied by $2\pi i$. On the other hand,

$$\begin{aligned}\int_{\partial\Pi} f(z)dz = \int_{c}^{c+\Omega} f(z)dz + \int_{c+\Omega}^{c+\Omega+\Omega'} f(z)dz + \\ + \int_{c+\Omega+\Omega'}^{c+\Omega'} f(z)dz + \int_{c+\Omega'}^{c} f(z)dz.\end{aligned}$$

Making the substitution

$$z = \zeta + \Omega'$$

in the third integral in the right-hand side, we get that

$$\int_{c+\Omega+\Omega'}^{c+\Omega'} f(z)dz = \int_{c+\Omega}^{c} f(\zeta+\Omega')d\zeta = \int_{c+\Omega}^{c} f(\zeta)d\zeta,$$

since $f(\zeta+\Omega') = f(\zeta)$.

Consequently, the third integral cancels with the first. Similarly, the second and fourth integrals cancel each other. Accordingly,

$$\text{sum of the residues} \equiv \int_{\partial\Pi} f(z)dz = 0, \tag{2.7}$$

as required.

□

Remark 2.4.2. In view of our definition, of two parallel sides only one can belong to the period parallelogram. Therefore, this result is valid also when the function has poles on the boundary of the parallelogram. It is only necessary to take all the poles lying *in* the parallelogram (and not only *inside* it).

Exercise 2.4.1. Prove the assertion in Remark 2.4.2

2.4.3. a-points. Let $a \in \mathbb{C}$ be any complex number. A point ζ is called an a-point of a function $f(z)$ if $f(\zeta) = a$.

Corollary 2.1. *Number of poles of a nonconstant elliptic function $f(z)$ in a period parallelogram is equal to the properly counted number of a-points, for an arbitrary a.*

Proof. To prove this statement it is suffices to substitute

$$\varphi(z) := \frac{f'(z)}{f(z)-a}$$

instead of $f(z)$. Indeed, let ζ_k be an arbitrarily chosen pole of $f(z)$. Then

$$f(z) = \frac{g_k(z)}{(z-\zeta_k)^{\nu_k}},$$

where $\nu_k \geq 1$ is the order of ζ_k and $g_k(z)$ is holomorphic near ζ_k function such that

$$g_k(\zeta_k) \neq 0. \tag{2.8}$$

Hence we have

$$\varphi(z) = \frac{f'(z)}{f(z)-a} = \frac{g_k'(z)(z-\zeta_k) - \nu_k g_k(z)}{g_k(z) - a(z-\zeta_k)^{\nu_k}} \cdot \frac{1}{z-\zeta_k} \equiv h_k(z) \cdot \frac{1}{z-\zeta_k}.$$

On the other hand, using (2.8) we obtain

$$h_k(\zeta_k) = -\nu_k,$$

which yields

$$\operatorname*{res}_{z=\zeta_k} \varphi(z) = -\nu_k.$$

Similarly, let u_k be an a-point of $f(z)$. Then we have near $z = u_k$

$$f(z) = a + (z-u_k)^{\mu_k} G_k(z),$$

where u_k is the order of u_k and $G_k(z)$ is holomorphic near ζ_k function satisfying (2.8) at $z = u_k$. Hence we obtain

$$\varphi(z) = \frac{G_k'(z)(z-u_k) + \mu_k G_k(z)}{G_k(z)} \cdot \frac{1}{z-u_k} \equiv H_k(z) \cdot \frac{1}{z-u_k}.$$

But $H_k(u_k) = \mu_k$, and we obtain

$$\operatorname*{res}_{z=u_k} \varphi(z) = \mu_k.$$

Applying Theorem 2.2 we get

$$\sum_{\text{"}a\text{-points"}} \mu_k - \sum_{\text{"poles"}} \nu_k = 0$$

where the sums are given over the a-points and the poles which are in a fixed period parallelogram, and the required assertion follows. ☐

Corollary 2.2 (Liouville's Theorem). *There does not exist a nonconstant elliptic function that is regular in a period parallelogram.*

Proof. Indeed, the number of poles of such a function would be equal to zero, and hence so would be the number of a-points for an arbitrary a, which is absurd. ☐

Definition 2.4.1. The number of poles in a period parallelogram, counting multiplicity, is called the *order* of the corresponding elliptic function.

Corollary 2.3. *The order of a nonconstant elliptic function f cannot be less than two.*

Proof. Excluding the trivial case $f \equiv const$ we can we can assume that at least one pole, say ζ_0, does exist in the period parallelogram of f. To prove the assertion it suffices to consider only the case when ζ_0 is a unique pole in the period parallelogram of the *first* order. Then we have near $z = \zeta_0$:

$$f(z) = \frac{g(z)}{z-\zeta_0}, \qquad g(\zeta_0) \neq 0.$$

But this implies

$$\operatorname*{res}_{z=\zeta_0} f(z) = g(\zeta_0) \neq 0,$$

which contradicts to (2.7). ☐

Thus, two types of elementary elliptic functions are conceivable a priori:

- a function of the **first** type has in a period parallelogram one pole of second order, with residue equal to zero;
- a function of the **second** type has two distinct poles of first order with residues differing only in sign.

Functions of both types will be constructed below. Before doing this we prove the residues theorem due to Liouville.

THEOREM 2.3. *Let $\alpha_1, \dots \alpha_m$ are the a-points of $f(z) \not\equiv$ const lying in a period parallelogram, with each point written as many times as its multiplicity has units. Further, $\beta_1, \dots \beta_m$ are the poles of the function, written according to the same principle. Then*

$$\sum_{k=1}^{m} \alpha_k \equiv \sum_{k=1}^{m} \beta_k \mod (\Omega, \Omega'). \tag{2.9}$$

PROOF. By Cauchy's theorem,

$$2\pi i \left(\sum_{k=1}^{m} \alpha_k - \sum_{k=1}^{m} \beta_k \right) = \int\limits_{\partial\Pi} \frac{zf'(z)}{f(z)-a}\, dz, \tag{2.10}$$

where Π is the period parallelogram such that $f(z)$ does not take the value a and does not have poles on $\partial\Pi$. Computing the contour integral as above, we find that

$$\begin{aligned} \int_{\partial\Pi} \frac{zf'(z)}{f(z)-a}\, dz = \left\{ \int_{c}^{c+\Omega} + \int_{c+\Omega}^{c+\Omega+\Omega'} + \int_{c+\Omega+\Omega'}^{c+\Omega'} + \int_{c+\Omega'}^{c} \right\} \frac{zf'(z)}{f(z)-a}\, dz = \\ = J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Here as above, we assume that the parallelogram Π has the vertices

$$c, \quad c+\Omega, \quad c+\Omega', \quad c+\Omega+\Omega'.$$

Making the substitution $z = \zeta + \Omega'$ in J_3, we get that

$$J_3 = \int_{c+\Omega}^{c} (\zeta + \Omega') \frac{\zeta f'(\zeta)}{f(\zeta)-a}\, d\zeta.$$

Therefore,

$$\begin{aligned} J_1 + J_3 = \Omega' \int\limits_{c+\Omega}^{c} \frac{\zeta f'(\zeta)}{f(\zeta)-a}\, d\zeta = \\ = \Omega' \left(\ln(f(c)-a) - \ln(f(c+\Omega)-a) \right); \end{aligned}$$

and since

$$f(c) - a = f(c+\Omega) - a,$$

it follows that $\ln(f(c)-a)$ differs from $\ln(f(c+\Omega)-a)$ only by an integer multiple of $2\pi i$; hence,

$$J_1 + J_3 = \Omega' \cdot 2n'\pi i.$$

It can be proved similarly that

$$J_2 + J_4 = \Omega \cdot 2n\pi i.$$

Consequently,

$$\sum_{k=1}^{m} \alpha_k = \sum_{k=1}^{m} \beta_k + n\Omega + n'\Omega',$$

and the theorem follows. □

We can reformulate the preceding property as follows

The sum of the a-points of $f(z)$ for an arbitrary a is congruent modulo the periods to the sum of the poles of the function if all the a-points and poles in a single period parallelogram are being considered.

Exercise 2.4.2. Prove (2.10) (the proof is similar to that in Corollary 2.1).

2.5. The Weierstrass function $\wp(z)$

2.5.1. Preliminaries. In this section we write $\Omega = 2\omega$ and $\Omega' = 2\omega'$, thus standing ω's for the half-periods.

We consider the series

$$\sum_{m,m'}{}' \frac{1}{|2m\omega + 2m'\omega'|^p}, \tag{2.11}$$

where the summation is over all integers m and m' except for the pair[1] $m = m' = 0$, and the numbers ω and ω' satisfy the assumption made above.

LEMMA 2.1. *The series (2.11) converges for $p > 2$ and diverges for $p > 2$ and diverges for $p \leq 2$.*

PROOF. We have here some regular system of points $2m\omega + 2m'\omega'$, from which the point 0 is removed. Let us first of all take the points

$$\pm 2\omega, \quad \pm(2\omega + 2\omega'), \quad \pm 2\omega', \quad \pm(2\omega - 2\omega') \tag{2.12}$$

of our regular system. These points are the vertices of four parallelograms coming together at point 0, and they form the first framing of the point 0. Suppose that the minimal distance from 0 to a vertex of the system (2.12) is d, and the maximal distance is D. Then the sum of the eight terms of the series (2.11) corresponding to the vertices (2.12) satisfies the inequalities

$$\frac{8}{D^p} \leq S_1 \leq \frac{8}{d^p}.$$

We now take the vertices of our regular system that belong to the second framing of 0. There will be 16 of these vertices, and the minimal and maximal distances from 0 to them will be $2d$ and $2D$, respectively. Therefore, the sum S_2 in the series (2.11) corresponding to these 16 vertices satisfies the inequalities

$$\frac{16}{(2D)^p} \leq S_2 \leq \frac{16}{(2d)^p}.$$

The nth framing will consist of $8n$ vertices, and the sum S_n corresponding to it satisfies the inequalities

$$\frac{8n}{(nD)^p} \leq S_n \leq \frac{8n}{(nd)^p}.$$

Convergence of our series (2.11) is equivalent to convergence of the series

$$S_1 + S_2 + \dots,$$

---

[1] This is indicated by the prime after the summation sign

and our assertion is an immediate consequence of the fact that

$$\frac{8}{D^p n^{p-1}} \leq S_n \leq \frac{8}{d^p n^{p-1}}.$$

□

Exercise 2.5.1. Show that the nth frame in the preceding proof has cardinality $8n$.

Corollary 2.4. *The series*

$$S(z) := -2 \sum_{m,m'} \frac{1}{(z - 2m\omega - 2m'\omega')^3} \tag{2.13}$$

converges absolutely and uniformly in each bounded region of the z-plane if the finite number of terms that become infinite there are removed. In particular, $S(z)$ is a meromorphic function whose only poles (which have order three) are $2m\omega + 2m'\omega'$.

Let $S(z)$ defined by (2.13). Then this function has periods 2ω and $2\omega'$. Indeed,

$$\begin{aligned} S(z + 2\omega) &= -2 \sum_{m,m'} \frac{1}{(z + 2\omega - 2m\omega - 2m'\omega')^3} \\ &\text{(setting } m - 1 = n) \\ &= -2 \sum_{n,m'} \frac{1}{(z - 2n\omega - 2m'\omega')^3}. \end{aligned}$$

But since the pair (n, m') runs through the same collection as the pair (m, m'), it follows that the right-hand side of the formula is also $S(z)$, and equality

$$S(z + 2\omega) = S(z)$$

is proved. It is proved in exactly the same way that

$$S(z + 2\omega') = S(z).$$

Moreover,

$$\begin{aligned} S(-z) &= -2 \sum_{m,m'} \frac{1}{(z + 2m\omega + 2m'\omega')^3} \\ &= \ 2 \sum_{n,n'} \frac{1}{(z - 2n\omega - 2n'\omega')^3} = -S(z). \end{aligned}$$

Here we take into account that the pairs (m, m') and (n, n') with $n = -m$ and $n' = -m'$ run through one and the same collection.

By integration we now introduce the function

$$\wp(z) := \frac{1}{z^2} + \int_0^z \left(S(u) + \frac{2}{u^3} \right) du.$$

Here it is assumed that the path of integration does not go through the vertices in the period network different from the origin $z = 0$. It follows from the special singular character of the poles of $S(z)$ that the latter integral has zero residues at the poles. Thus, $\wp(z)$ is well defined and

$$\wp'(z) = S(z), \tag{2.14}$$

and, on the other hand, termwise integration yields

$$\wp(z) = \frac{1}{z^2} + \sum_{m,m'}{}' \left\{ \frac{1}{(z + 2\omega - 2m\omega - 2m'\omega')^2} - \frac{1}{(2\omega - 2m\omega + 2m'\omega')^2}. \right\} \tag{2.15}$$

Since $S(z)$ is an odd function, it follows that $\wp(z)$ is an *even* function. This circumstance can also be obtained easily with the help of representation (2.15).

Further, since $S(z)$ has period 2ω, (2.14) implies that

$$\wp'(z + 2\omega) = \wp'(z),$$

and hence,

$$\wp(z + 2\omega) = \wp(z) + c, \tag{2.16}$$

where c is a constant.

It follows from the expansion (2.15) that the only poles of $\wp(z)$ are the points $2m\omega + 2m'\omega'$; therefore, $\wp(z)$ is finite at the points ω and ω'. But since the substitution $z = -\omega$ in (2.16) gives

$$\wp(\omega) = \wp(-\omega) + c,$$

it follows from the evenness of $\wp(z)$ that c has the value 0, i.e.

$$\wp(z + 2\omega) = \wp(z).$$

It can be verified similarly that

$$\wp(z + 2\omega') = \wp(z).$$

Thus, we obtain

THEOREM 2.4. *The Weierstrass function $\wp(z)$ is an elliptic function of second order with periods 2ω and $2\omega'$.*

FIGURE 4. KARL WEIERSTRASS (1815-1897)

2.5.2. The differential equation of the function $\wp(z)$. In a neighborhood of the point $z = 0$ the function $\wp(z)$ has the form

$$\begin{gathered}\wp(z) = \frac{1}{z^2} + 3z^2 \sideset{}{'}\sum_{m,m'} \frac{1}{(2m\omega + 2m'\omega')^4} + \\ + 5z^4 \sideset{}{'}\sum_{m,m'} \frac{1}{(2m\omega + 2m'\omega')^6} + \ldots\end{gathered} \tag{2.17}$$

Exercise 2.5.2. Prove the preceding representation. Show, that the rest of the coefficients are equal to the series

$$\sum_{k=1}^{\infty} A_k z^{2k} \sideset{}{'}\sum_{m,m'} \frac{1}{(2m\omega + 2m'\omega')^{2k+2}},$$

where A_k are some numerical factors. Find A_3.

We adopt the notation

$$\sideset{}{'}\sum_{m,m'} \frac{1}{(2m\omega + 2m'\omega')^4} = \frac{g_2}{60}, \qquad \sideset{}{'}\sum_{m,m'} \frac{1}{(2m\omega + 2m'\omega')^6} = \frac{g_3}{140}.$$

In this notation

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20} z^2 + \frac{g_3}{28} z^4 + \ldots. \tag{2.18}$$

Then we have

$$\wp'(z) = -\frac{2}{z^3} + \frac{g_2}{10} z + \frac{g_3}{7} z^3 + \ldots.$$

Therefore, squaring we have

$$(\wp'(z))^2 = \frac{4}{z^6}(1 - \frac{g_2}{10} z^4 - \frac{g_3}{7} z^6 + \ldots)$$

and

$$(\wp'(z))^3 = \frac{1}{z^6}(1 + \frac{3g_2}{20} z^4 + \frac{3g_3}{28} z^6 + \ldots)$$

In view of these expansions and (2.18),

$$\wp'^2(z) - 4\wp^3(z) + g_2\wp(z) = -g_3 + Az^2 + Bz^4 + \ldots.$$

The left-hand side is an elliptic function with periods 2ω and $2\omega'$. Only points $2\omega m + 2\omega' m'$ can be poles of it. But since this function is regular and equal to $-g_3$ at the point $z = 0$ (as the above formula shows), it is regular in every period parallelogram with $z = 0$ as an interior point, and hence it is constant by Liouville's theorem. Accordingly, we have obtained the relation

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3. \tag{2.19}$$

In other words, $\wp(z)$ satisfies the differential equation

$$f'^2 = 4f^3 - g_2 f - g_3.$$

Remark 2.5.1. It is easy to see that any function of the form

$$\wp(\pm z + c)$$

also satisfies (2.19), where c is an arbitrary constant. But since $\wp$ is an even function of z we can drop the sign ±1 before z actually.

Equation (2.19) enables us to express all the derivatives of $\wp(z)$ in terms of $\wp(z)$ and $\wp'(z)$; for example,

$$\begin{aligned} \wp'' &= 6\wp^2 - \frac{1}{2}g_2, \\ \wp''' &= 12\wp\wp', \end{aligned} \tag{2.20}$$

etc. Let

$$4\zeta^3 - g_2\zeta - g_3 = 4(\zeta - e_1)(\zeta - e_2)(\zeta - e_3). \tag{2.21}$$

Then

$$\begin{aligned} e_1 + e_2 + e_3 &= 0, \\ e_1e_2 + e_2e_3 + e_1e_3 &= -\frac{1}{2}g_2, \\ e_1e_2e_3 &= -\frac{1}{4}g_3, \end{aligned} \tag{2.22}$$

and we have

$$e_1^2 + e_2^2 + e_3^2 = (e_1 + e_2 + e_3)^2 - 2(e_1e_2 + e_2e_3 + e_1e_3) = \frac{1}{2}g_2.$$

Observing that $\wp'(z)$ ia an odd function and setting $z = -\omega$ in the equality

$$\wp'(z + 2\omega) = \wp'(z),$$

we find that $\wp'(\omega) = 0$ (since $\wp'(\omega)$ is finite). It can be shown similarly that

$$\wp'(\omega') = 0, \qquad \wp'(\omega + \omega') = 0.$$

We see that the points ω, ω' and $\omega+\omega'$ are *simple* zeroes of the first derivative $\wp'(z)$. Indeed, we know that $\wp(z)$ is a second order elliptic function and it follows that $\wp'(z)$ has the third order.

COROLLARY 2.5. *The quantities $\wp(\omega)$, $\wp(\omega')$ and $\wp(\omega + \omega')$ are all distinct.*

PROOF. Indeed, is, for example $\wp(\omega) = \wp(\omega')$, then the second order elliptic function

$$f(z) := \wp(z) - \wp(\omega)$$

would have the two second order zeroes ω and $\omega + \omega'$, which is impossible. □

Exercise 2.5.3. Finish the proof of the latter corollary.

In view of (2.19) the quantities $\wp(\omega)$, $\wp(\omega')$ and $\wp(\omega+\omega')$ coincides with the roots of the polynomial (2.21). Therefore, the numbers e_1, e_2 and e_3 are all distinct.

It is frequently more convenient to use the notation

$$\begin{aligned} 2\omega_1 &= 2\omega, \\ 2\omega_2 &= -2\omega - 2\omega', \\ 2\omega_3 &= 2\omega', \end{aligned}$$

so that

$$\omega_1 + \omega_3 + \omega_3 = 0.$$

Further, one sets

$$\wp(\omega_k) = e_k \qquad (k = 1, 2, 3).$$

It is useful to remark that formulas (2.22) lead to the following representation of the *discriminant*

$$g_2^3 - 27g_3^2 = 16(e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2, \tag{2.23}$$

as well as the equality

$$\frac{3}{2}g_2 = (e_1 - e_2)^2 + (e_2 - e_3)^2 + (e_3 - e_1)^2. \tag{2.24}$$

Remark 2.5.2. Given a polynomial

$$P(z) := a_0 + a_1 z + \dots a_n z^n,$$

the value

$$D(P) := a_n^{2n-2} \prod_{i<j} (z_i - z_j)^2$$

is called the *discriminant* of P provided that z_j, $1 \le j \le n$ are the roots of P accounted with their multiplicities.

An important formula is the definition of the main modular form given as follows

$$J \equiv \frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{[(e_1 - e_2)^2 + (e_2 - e_3)^2 + (e_3 - e_1)^2]^3}{54(e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2}. \tag{2.25}$$

Exercise 2.5.4. Prove (2.23) and (2.24).

Exercise 2.5.5. Prove that $\wp$ has the following representation near $z = 0$:

$$\wp(z) = z^{-2} + \sum_{k=1}^{\infty} c_{2k} z^{2k},$$

where

$$c_2 = \frac{g_2}{20}, \quad c_4 = \frac{g_3}{28}, \quad c_6 = \frac{g_2^2}{2^4 \cdot 3 \cdot 5^2}, \quad c_8 = \frac{3g_2 g_3}{2^4 \cdot 5 \cdot 7 \cdot 11}, \dots.$$

2.5.3. The addition-theorem for $\wp(z)$.

The function $\wp(z)$ possesses the following addition-theorem.

Theorem 2.5. *Let u, v and w are complex numbers different from the poles of $\wp(z)$ such that*

$$u + v + w = 0. \tag{2.26}$$

Then

$$\det \begin{pmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{pmatrix} = 0. \tag{2.27}$$

Proof. We notice that

$$-w = u + v := \zeta.$$

The case

$$u \equiv \pm v \mod (\omega, \omega') \tag{2.28}$$

is trivial and leads to (2.27) immediately.

Let now the last congruence is false. Then consider the following linear system

$$\wp'(u) = A\wp(u) + B, \qquad \wp'(v) = A\wp(v) + B,$$

which determines A and B uniquely. Indeed, all the coefficients are finite values and the discriminant

$$\det \begin{pmatrix} \wp(u) & 1 \\ \wp(v) & 1 \end{pmatrix} = \wp(u) - \wp(v)$$

is non-zero since otherwise we have (2.28).

Now consider

$$f(z) := \wp'(z) - A\wp(z) - B$$

as a function of z. Then it has a triple pole at $z = 0$. Consequently, by the Liouville theorem it has three zeroes (counting with their multiplicities). On the other hand, the sum of these zeroes z_k is congruent the sum of poles, i.e. the zero period. Other words,

$$z_1 + z_2 + z_3 \equiv 0 \mod (\omega, \omega').$$

But $z_1 = u$, $z_2 = v$ are two zeroes which implies that

$$z_3 = -(u + v) = w$$

is the third zero. Thus, we have

$$\wp'(w) - A\wp(w) - B = 0.$$

Combining these three equalities into a linear system

$$\begin{pmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we conclude that its determinant is zero and the theorem follows. □

Since the derivatives occurring in the determinant can be expressed algebraically in terms of $\wp(u)$, $\wp(v)$ and $\wp(w)$ respectively this result really expresses $\wp(w) = \wp(u+v)$ algebraically in terms of $\wp(u)$ and $\wp(v)$.

2.5.4. The addition-theorem for $\wp(z)$, II. Another form of the previous addition-theorem can be obtained as follows. Retaining the above notations, we see that the values of z which make $\wp'(z) - A\wp(z) - B$ vanish, are congruent to one of the points u, v and $-u - v$.

Hence,

$$\wp'^2(z) - (A\wp(z) + B)^2$$

vanishes when z is congruent to any of the points u, v and $-u - v$. And so, substituting the derivative from (2.19) we find that the function

$$4\wp^3(z) - A^2\wp^2(z) - (2AB + g_2)\wp(z) - (B^2 + g_3)$$

vanishes when $\wp(z)$ is equal to one of

$$\wp(u), \quad \wp(v) \quad \text{or} \quad \wp(w).$$

For general values of u and v, $\wp(u)$ and $\wp(u + v)$ are unequal and so they are all the roots of the equation

$$4\zeta^3 - A^2\zeta^2 - (2AB + g_2)\zeta - (B^2 + g_3) = 0.$$

Consequently, by the ordinary formula for the sum of the roots of a cubic equation,

$$\wp(u) + \wp(v) + \wp(u + v) = \frac{1}{4}A^2,$$

hence,

$$\wp(u+v) = \frac{1}{4}\left\{\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right\}^2 - \wp(u) - \wp(v), \tag{2.29}$$

on solving the equations by which A and B were defined.

The latter formula expresses $\wp(u+v)$ explicitly in terms of functions of u and v.

The forms of the addition-theorem which have been obtained are both nugatory when $u = v$. But (2.29) is valid in the case of any given value of u, for general values of v. Taking the limiting form of the latter result when v approaches to u, we have

$$\lim_{v\to u} \wp(u+v) = \lim_{v\to u} \frac{1}{4}\left\{\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right\}^2 - \wp(u) - \lim_{v\to u} \wp(v).$$

From this equation, we see that if $2v$ is not a period we have

$$\wp(2u) = \lim_{h\to 0} \frac{1}{4}\left\{\frac{\wp'(u)-\wp'(u+h)}{\wp(u)-\wp(u+h)}\right\}^2 - 2\wp(u) =$$

$$= \frac{1}{4}\left\{\frac{\wp''(u)}{\wp'(u)}\right\}^2 - 2\wp(u),$$

unless $2u$ is a period. The result is called the *duplication formula* for $\wp$.

Exercise 2.5.6. Prove that

$$\frac{1}{4}\left\{\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}\right\}^2 - \wp(z) - \wp(u)$$

as a function of z has no singularities at points congruent with $z = 0, \pm u$; and, by making use of Liouville's theorem, deduce the addition-theorem.

Exercise 2.5.7. Apply the process indicated in Exercise 2.5.6 to the function

$$\det\begin{pmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{pmatrix}$$

and deduce the addition-theorem.

Exercise 2.5.8. Show that

$$\wp(u+v) + \wp(u-v) = \frac{[2\wp(u)\wp(v) - \frac{1}{2}g_2](\wp(u)+\wp(v)) - g_3}{(\wp(u)-\wp(v))^2}$$

(**Hint:** Apply the addition formula (2.29) to the left-hand side, and consequently the main expression for the derivative $\wp'^2$).

2.5.5. The addition of a half-period. Let $u = z$ and $v = \omega$ in (2.29). Then we obtain

$$\wp(z+\omega) + \wp(z) + \wp(\omega) = \frac{1}{4}\left\{\frac{\wp'(z)-\wp'(\omega)}{\wp(z)-\wp(\omega)}\right\}^2,$$

and so, since

$$\wp'^2(z) = 4\prod_{k=1}^{3}(\wp(z) - e_k),$$

(where e_k are defined by (2.21)) we have

$$\wp(z+\omega) = \frac{1}{4}\frac{(\wp(z)-e_2)(\wp(z)-e_3)}{\wp(z)-e_1} - \wp(z) - e_1,$$

i.e.

$$\wp(z+\omega) = \frac{(e_1-e_2)(e_1-e_3)}{\wp(z)-e_1} + e_1, \tag{2.30}$$

on using $e_1+e_2+e_3=0$. Thus, (2.30) expresses $\wp(z+\omega)$ in terms of $\wp(z)$.

Exercise 2.5.9. Show that

$$\wp(\omega/2) = e_1 \pm \sqrt{(e_1-e_2)(e_1-e_3)}.$$

Exercise 2.5.10. Show that

$$\wp(u+v) - \wp(u-v) = -\frac{\wp'(u)\wp'(v)}{[\wp(u)-\wp(v)]^2}.$$

2.6. Modular forms

2.6.1. Lattices. As we know, two numbers 2ω and $2\omega'$ whose ratio

$$\tau := \frac{\omega'}{\omega} \tag{2.31}$$

has nonzero imaginary part give rise to a regular system, or a *lattice*, of points on the complex plane. The same regular system of points can be obtained by starting with certain other pairs of numbers.

Namely, define the lattice

$$L(\omega,\omega') := \{z = m\omega + m'\omega' : \quad m, m' \in \mathbb{Z}\}.$$

LEMMA 2.2. *$L(\omega,\omega') = L(\omega_1,\omega_1')$ if and only if there exist an integer valued matrix*

$$H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with property $\det H = \pm 1$ *such that*

$$\begin{pmatrix} \omega_1 \\ \omega_1' \end{pmatrix} = H \begin{pmatrix} \omega \\ \omega' \end{pmatrix}. \tag{2.32}$$

Moreover, in this case

$$\operatorname{sign}(\operatorname{Im}\tau_1) = \det H \cdot \operatorname{sign}(\operatorname{Im}\tau). \tag{2.33}$$

REMARK 2.6.1. An integer valued matrix H is said to be *unimodular* if its determinant equals 1.

PROOF. First we suppose that the pair (ω_1,ω_1') satisfies the hypotheses of the lemma, i.e. there exists a matrix H with integer coefficients and $\det H = \pm 1$ such that (2.32) holds. Let ζ be any point of the lattice $L(\omega,\omega')$. Then

$$\zeta = m\omega_1 + m'\omega_1' = m(a\omega + b\omega') + m'(c\omega + d\omega') = M\omega + M'\omega'$$

where $M = am + cm'$ and $M' = bm + dm'$ are integers. Thus, $\zeta \in L(\omega,\omega')$, i.e.

$$L(\omega_1,\omega_1') \subset L(\omega,\omega').$$

Since $\det H = 1$ we have for the inverse relation

$$\begin{pmatrix} \omega \\ \omega' \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_1' \end{pmatrix}.$$

and it follows by a similar argument that $L(\omega_1, \omega_1') \subset L(\omega, \omega')$, which implies

$$L(\omega_1, \omega_1') = L(\omega, \omega'). \tag{2.34}$$

Let now (2.34) holds. Since ω_1, ω_1' are linear independent over $\mathbb{R}$ we can find a linear transformation H with real coefficients a, b, c and d such that (2.32) holds. Let us show that these coefficients are integers. Indeed, in view of (2.34) we have for all integer m and m':

$$M = am + cm' \in \mathbb{Z}, \qquad M' = bm + dm' \in \mathbb{Z}. \tag{2.35}$$

Letting $m' = m - 1 = 0$ we get $M = a \in \mathbb{Z}$, and same argument implies that $b, c, d \in \mathbb{Z}$.

Now we claim that $\delta := \det H = ad - bc = \pm 1$. Indeed, using (2.35) and (2.34) we see that the numbers

$$m = \frac{1}{\delta}(dM - cM'), \qquad m' = \frac{1}{\delta}(-bM + aM'),$$

are integer for any choice of integers M and M'. It follows as above that the coefficients of the corresponding linear transform

$$H' := \frac{1}{\delta} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

are integers. Thus, the determinant is an integer too. But we have

$$\det H' = \frac{1}{\delta} \in \mathbb{Z},$$

and it follows that $\delta = \pm 1$.

It remains only to establish (2.33). We have

$$\tau_1 = \frac{c\omega + d\omega'}{a\omega + b\omega'},$$

and it follows that

$$\begin{aligned} \operatorname{Im} \tau_1 &= \operatorname{Im} \frac{(c\omega + d\omega')(a\bar{\omega} + b\bar{\omega}')}{|a\omega + b\omega'|^2} = \operatorname{Im} \frac{bc\,\omega\bar{\omega}' + ad\,\omega'\bar{\omega}}{|a\omega + b\omega'|^2} = \\ &= \frac{bc}{|a\omega + b\omega'|^2} \operatorname{Im} \omega\bar{\omega}' - \frac{ad}{|a\omega + b\omega'|^2} \operatorname{Im} \omega\bar{\omega}' = \\ &= \frac{\delta}{|a\omega + b\omega'|^2} \operatorname{Im} \omega'\bar{\omega} = \frac{\delta|\omega|^2}{|a\omega + b\omega'|^2} \operatorname{Im} \frac{\omega'}{\omega} \end{aligned}$$

and the desired relation follows. □

Definition 2.6.1. The pairs $(2\omega, 2\omega')$ and $(2\omega_1, 2\omega_1')$ are said to be equivalent if they are linked by a unimodular transformation (2.32).

2.6.2. Modular substitutions. The quantities

$$g_2 = g_2(\omega,\omega') := 60\sum_{m,m'}{}' \frac{1}{(2m\omega+2m'\omega')^4},$$

$$g_3 = g_2(\omega,\omega') := 140\sum_{m,m'}{}' \frac{1}{(2m\omega+2m'\omega')^6},$$

are relative invariant of the polynomial in $\wp(z)$ that represents $\wp'^2(z)$. We now regard these quantities as functions of the pair $(2\omega, 2\omega')$. As is easily seen. they do not change if instead of pair $(2\omega, 2\omega')$ we take another pair generating the same lattice of points. In particular, g_2 and g_3 do not change upon passing from the pair $(2\omega, 2\omega')$ to an equivalent pair $(2\omega_1, 2\omega_1')$.

On the other hand, it follows immediately from the definition of g_2 and g_3 that

$$g_2(t\omega, t\omega') = t^{-4} g_2(\omega,\omega'),$$

$$g_3(t\omega, t\omega') = t^{-6} g_3(\omega,\omega').$$

Replacement of the pair $(2\omega, 2\omega')$ by $(2t\omega, 2t\omega')$ corresponds to the transition from the original lattice to a similar lattice. As we see, g_2 and g_3 are not invariant under such transformations. However, the quantity (2.25)

$$J \equiv \frac{g_2^3}{g_2^3 - 27g_3^2}$$

clearly remains unchanged not only under the transition form the pair $(2\omega, 2\omega')$ to an equivalent pair $(2t\omega, 2t\omega')$, but also under the the transition from the original lattice to a similar lattice: $(2\omega, 2\omega') \to (2t\omega, 2t\omega')$.

This quantity J, called above an absolute invariant, is thus a function of a single variable, namely of the ratio

$$\tau = \frac{\omega'}{\omega},$$

and it has the following property which easily follows from Lemma 2.2:

Proposition 2.5. *For any integers α, β, γ and δ such that*

$$\alpha\delta - \beta\gamma = 1, \tag{2.36}$$

the equality holds

$$J\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = J(\tau)$$

Remark 2.6.2. The linear substitution

$$\tau' = \frac{\alpha\tau+\beta}{\gamma\tau+\delta},$$

where the coefficients connected by the relation (2.36), is called a *modular substitution.* It is also worthy mention that

$$\frac{d\tau'}{d\tau} = \frac{\alpha\delta-\beta\gamma}{(\gamma\tau+\delta)^2}.$$

An analytic function that is invariant under modular substitutions is called a *modular function.*

It will be proved below that $J(\tau)$ is an analytic function. Therefore, $J(\tau)$ is an modular function. As for the invariants g_2 and g_3, which are not functions of τ, they called modular forms of ω and ω'.

Modular substitutions will be denoted by the letters $S, T, \ldots$. For example, if

$$\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta},$$

then we write

$$\tau' = S\tau, \tag{2.37}$$

and

$$S = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} -\alpha & -\beta \\ -\gamma & -\delta \end{pmatrix}\},$$

where we use the brackets to emphasize that we do not distinguish two matrices S and $-S$ (since they produce the same modular substitution).

The identity substitution $\tau' = \tau$ is also modular and will be denoted by I:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The expression (2.37) underscores that τ' is regarded as the result of applying some operation to τ. If

$$\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta} = S(\tau),$$

then

$$\tau = \frac{-\delta\tau' + \beta}{\gamma\tau - \alpha}.$$

The corresponding substitution S^{-1}

$$S^{-1} := \begin{bmatrix} -\delta & \beta \\ \gamma & -\alpha \end{bmatrix}$$

is also modular and is called the inverse of S.

Exercise 2.6.1. Prove that if $S(\tau)$ and $T(\tau)$ are two modular substitutions then its composition $S \circ T(\tau)$ has the matrix representation $S \cdot T$.

The substitution $S \circ T$ is commonly called the product of S and T.

Exercise 2.6.2. Find the matrix representation of the *commutator* of two modular substitutions $S(\tau)$ and $T(\tau)$:

$$[S, T] := S \circ T \circ S^{-1} \circ T^{-1}.$$

In particular, deduce that $S \circ T \neq T \circ S$ in general.

With respect to this multiplication operation *the collection of all modular substitutions form a group.*

Exercise 2.6.3. Prove the preceding property.

The function $J(\tau)$ is invariant under this group of transformations. It is often necessary to consider other groups of linear fractional transformations. An analytic function that is invariant under such a transformation group is always called an automorphic function. Thus, $J(\tau)$ is an example of an automorphic function. Periodic functions are also simplest examples of automorphic functions.

2.6.3. Fundamental regions of the group Σ. It suffices to study a doubly periodic function in some period parallelogram. The group of substitutions with respect to which a doubly periodic function is invariant is generated by the two basic substitutions:

$$\begin{aligned} S: &\quad \widetilde{u} = u + 2\omega, \\ S': &\quad \widetilde{u} = u + 2\omega', \end{aligned} \tag{2.38}$$

i.e., every substitutions of this group is the result of composing (multiplying) these substitutions.

Each of the basic substitutions S and S' connects a pair of opposite sides of a period parallelogram. Applying to this parallelogram all the substitutions in the group, we get an infinite set of congruent parallelograms covering the whole complex plane once.

For each point z of the plane, a period parallelogram contains one and only one point z' that is congruent to u modulo the periods, or, in other words, *is equivalent to z with respect to the group.* Therefore, a period parallelogram is a *fundamental region* of the group under consideration.

We turn now to the modular function $J(\tau)$. The group of modular substitutions is denoted by Σ ($J(\tau)$ is invariant with respect to Σ).

We show that Σ is generated by the two basic substitutions

$$\begin{aligned} S &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \widetilde{\tau} = \tau + 1, \\ T &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \widetilde{\tau} = -1/\tau. \end{aligned} \tag{2.39}$$

Let

$$V = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

be an arbitrary substitution in Σ.

Using the rule for multiplying substitutions, we get that

$$VT = \begin{bmatrix} \beta & -\alpha \\ \delta & -\gamma \end{bmatrix},$$

and also that

$$VS = \begin{bmatrix} \alpha & \beta + \alpha \\ \gamma & \delta + \gamma \end{bmatrix}, \qquad VS^{-1} = \begin{bmatrix} \alpha & \beta - \alpha \\ \gamma & \delta - \gamma \end{bmatrix}$$

and in general, for any integer n

$$VS^{-n} = \begin{bmatrix} \alpha & \beta - n\alpha \\ \gamma & \delta - n\gamma \end{bmatrix}.$$

We successively apply two operations, namely, multiplication of a substitution (from the right) by some power of the substitution S, and multiplication by T, and we show that by starting from an (arbitrary) substitution V, we can arrive in this way at the substitution

$$VS^{-n}TS^{-m}T\cdots TS^{-k} = \begin{bmatrix} \alpha^* & 0 \\ \gamma^* & \delta^* \end{bmatrix} \tag{2.40}$$

Indeed, if $\beta = 0$, then the original substitution satisfies the required property. Assuming that $\beta \neq 0$, we determine an integer n such that $|\beta - n\alpha| < \alpha$. After n has found, we consider the substitution

$$V_1 = VS^{-n} = \begin{bmatrix} \alpha & \beta - n\alpha \\ \gamma & \delta - n\gamma \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix};$$

here $|\beta_1| < |\alpha_1|$. If $\beta_1 = 0$, then it is the desired substitution. But if $\beta_1 \neq 0$, then we multiply the substitution V_1 by TS^{-m}, where is an integer, and as a result we get the substitution

$$V_2 = V_1TS^{-m} = \begin{bmatrix} \beta_1 & -\alpha_1 - m\beta_1 \\ \delta_1 & -\gamma_1 - m\delta_1 \end{bmatrix} = \begin{bmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{bmatrix},$$

where m is chosen so that $|\alpha_1 + m\beta_1| < |\beta_1|$. Thus, for V_2 we have $|\beta_2| < |\alpha_2|$. But since $|\alpha_2| = |\beta_1|$, it follows that

$$|\beta_2| < |\beta_1|.$$

Continuing these operations, we get substitutions

$$V_k = \begin{bmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{bmatrix}, \quad k = 1, 2, 3, \dots$$

with $|\beta_k| > |\beta_{k+1}|$. Since β_k are integers, we arrive finitely many operations at a substitution of the form (2.40) we need with $\beta_m = 0$.

But since this substitution is modular, it follows that $\alpha^*\delta^* = 1$, hence we can assume that $\alpha^* = \delta^* = 1$. Consequently,

$$VS^{-n}TS^{-m}T\cdots TS^{-k} = \begin{bmatrix} 1 & 0 \\ -p & 1 \end{bmatrix},$$

where p is some integer. But

$$\begin{bmatrix} 1 & 0 \\ -p & 1 \end{bmatrix} = TS^pT;$$

therefore

$$VS^{-n}TS^{-m}T\cdots TS^{-k} = TS^pT,$$

whence we derive

$$V = TS^pTS^kT\cdots TS^mTS^n.$$

Thus, it is proved that Σ is generated by the substitutions (2.39).

To get a fundamental region of the group Σ, we construct in the upper half-plane a triangle with sides

$$\operatorname{Re}\tau = -1/2, \qquad \operatorname{Re}\tau = 1/2, \qquad |\tau| = 1.$$

We define the region D to be the collection of all points lying inside this triangle, along with the points lying on the left side $\operatorname{Re}\tau = -1/2$, and the points lying on the circle $|\tau| = 1$ such that

$$-1/2 \leq \operatorname{Re}\tau \leq 0.$$

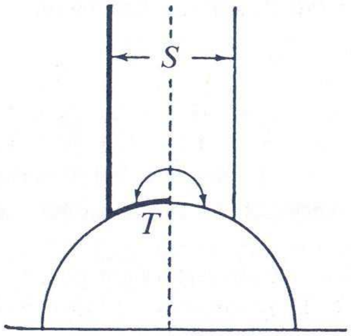


Figure 5. The fundamental region

Thus, D cab be regarded as a quadrangle (Figure 5), with only two of the four sides included (the thick lines in the figure).

The basic substitutions (2.39) connect the pairs of sides of the quadrangle; namely, S connects the vertical sides, and T carries the left arc of the circle into the right arc, as pictured in Figure 5.

Definition 2.6.2. Two points τ and τ' is called *equivalent* if Σ contains a substitution V such that $\tau' = V\tau$. A domain $E \subset \mathbb{C}$ is said to be a *fundamental region* if for every point of the upper half-plane there is one and only one equivalent point τ' in E.

Theorem 2.6. *Let D be the above set. Then D is a fundamental region of the group Σ.*

Proof. Let τ be a point with $\operatorname{Im}\tau > 0$. Take the pair $(1, \tau)$ of numbers and consider the lattice on the plane generated by this pair. Let all the points $m\tau + n$ in this lattice be numbered in order of nondecreasing modulus $m\tau + n$. We get a sequence

$$0, w_1, w_2, w_3, \ldots \qquad (w_{2k} = -w_{2k-1}). \tag{2.41}$$

In this sequence we take the first point that does not lie on the line joining the origin 0 to the point w_1. Let this be the point w_k, so that

$$|w_k| \geq |w_1|. \tag{2.42}$$

Both the points $w_k \pm w_1$, which occur in the sequence (2.41), have in it indices greater than k, because these points do not lie on the indicated line. Therefore,

$$|w_k \pm w_1| \geq |w_k|. \tag{2.43}$$

We can assume that

$$\operatorname{Im}\frac{w_k}{w_1} > 0,$$

since the last expression in non-zero and if it is negative we could replace w_1 by $-w_1$ (in other words, interchange the elements w_1 and w_2). As follows from its construction, the closed parallelogram with vertices 0, w_1, $w_1 + w_k$ and w_k does not contain points of the lattice other than its vertices. Therefore, every point of the regular system can be represented in the form $mw_1 + m'w_k$ with integers m and m'. Hence, the pair (w_1, w_k) is equivalent to the pair $(1, \tau)$.

Let $w_k = \alpha\tau + \beta$ and $w_1 = \gamma\tau + \delta$, where α, β, γ and δ are integers. Here

$$\alpha\delta - \beta\gamma = 1$$

since $\operatorname{Im}\tau > 0$ and $\operatorname{Im}(w_k/w_1) > 0$. We now let $w_k/w_1 = \widetilde{\tau}$, so that

$$\widetilde{\tau} = \frac{\alpha\tau + \beta}{\gamma\tau + \delta} = V\tau,$$

where $V \in \Sigma$.

It follows then from (2.42) and (2.43) that

$$|\widetilde{\tau}| \geq 1, \quad |\widetilde{\tau} + 1| \geq |\widetilde{\tau}|, \quad |\widetilde{\tau} - 1| \geq |\widetilde{\tau}|.$$

Consequently, the point $\widetilde{\tau}$ lies in the closed "triangle" with sides

$$\operatorname{Re}\tau = -1/2, \quad \operatorname{Re}\tau = 1/2, \quad |\tau| = 1 \qquad (\operatorname{Im}\tau > 0).$$

It turns out that

- if $\operatorname{Re}\widetilde{\tau} \neq 1/2$, $|\widetilde{\tau}| > 1$, or $-1/2 \leq \operatorname{Re}\widetilde{\tau} \leq 0$, $|\widetilde{\tau}| = 1$, then the point $\widetilde{\tau}$ lies in D and thus is the desired point: $\tau' = \widetilde{\tau}$;
- if $\operatorname{Re}\widetilde{\tau} = 1/2$, $|\widetilde{\tau}| > 1$, then $\tau' = \widetilde{\tau} - 1$ is the desired point;
- finally, if $0 < \operatorname{Re}\widetilde{\tau} \leq 1/2$, $|\widetilde{\tau}| = 1$, then $\tau' = -1/\widetilde{\tau}$ is the desired point.

Thus, it is proved that for every point τ in the upper half-plane there is an equivalent point $\tau' \in D$.

We now prove that D does not contain equivalent points. Assume the contrary, and let τ and τ' be the equivalent points in D. They cannot be connected by a transformation S^k nor by the transformation T. Hence,

$$\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \qquad (\neq -\frac{1}{\tau}),$$

and $\gamma > 0$. Since

$$\tau' - \frac{\alpha}{\gamma} = -\frac{\alpha\delta - \beta\gamma}{\gamma(\gamma\tau + \delta)},$$

it follows that

$$\tau' - \frac{\alpha}{\gamma} = -\frac{1}{\gamma(\gamma\tau + \delta)},$$

which yields

$$\left|\tau' - \frac{\alpha}{\gamma}\right| \cdot \left|\tau' + \frac{\delta}{\gamma}\right| = \frac{1}{\gamma^2}. \tag{2.44}$$

By assumption, both points τ and τ' lie in D, and the numbers $|\tau' - \alpha/\gamma|$ and $|\tau' + \delta/\gamma|$ represent the distances between these points and some points on the real axis. Consequently, each of these numbers is $\geq \sqrt{3}/2$. This implies $\gamma = 1$, and relation (2.44) takes the form

$$|\tau' - \alpha| \cdot |\tau' + \delta| = 1. \tag{2.45}$$

The distance from a point in D to an integer point of the real axis is ≥ 1. Therefore, it follows from (2.45) that

$$|\tau' - \alpha| = |\tau' + \delta| = 1.$$

hence, $\alpha = 0$ or -1, and $\delta = 0$ or 1. For $\alpha = -1$

$$\tau' = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

and for $\delta = 1$ we have

$$\tau = -\frac{1}{2} + i\frac{\sqrt{3}}{2}. \tag{2.46}$$

Consequently, the possibility $\alpha = -1$ and $\delta = 1$ is excluded. But if $\alpha = 0$, then $\delta \neq 0$, because $|\alpha| + |\delta| \neq 0$. Therefore, for $\alpha = 0$ we must have that $\delta = 1$ and $\beta = -1$ (since $\gamma = 1$), i.e.,

$$\tau' = -\frac{1}{1+\tau},$$

which, by (2.46), implies that

$$\tau' = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \tau.$$

Hence, this possibility is also excluded, and it can be established similarly that the equalities $\alpha = -1$ and $\delta = 0$ are excluded. The theorem is proved completely. □

Bibliography

[1] Akheizer, N.I. Elements of the Theory of Elliptic Functions. Providence, RI: Amer. Math. Soc., 1990. 237 p.

[2] Jonathan M. Borwein and Peter B. Borwein, Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity. New York: Wiley, 1987.

[3] B. C. Carlson, Algorithms involving arithmetic and geometric means. *MAA Monthly*, **78**(1971), 496-505.

[4] Arthur Cayley, An Elementary Treatise on Elliptic Functions, 2nd ed., 1895. Reprinted by Dover Pub., New York , 1961. (see also `http://historical.library.cornell.edu/math/math_C.html`)

[5] K. Chandrasekharan, Elliptic Functions, Springer-Verlag, 1985.

[6] David A. Cox, The arithmetic-geometric meanof Gauss. *L'Enseignement Mathématique*, **30**(1984), 275-330.

[7] Einar Hille, Analytic Function Theory, Vol. 2, Ginn & Co., Introductions to Higher Mathematics. 1962.

[8] Lawden, Derek F. Elliptic Functions and Applications. New York: Springer-Verlag, 1989. 334 p.

[9] E. H. Lockwood, A Book of CURVES, Cambridge at the University Press, 1961.

[10] V.H. Moll, P.A. Neill, J.L. Nowalsky and L. Solanilla, A property of Eulers elastic curve. *Elem. Math.* **55**(2000), 156–162.

[11] Andrew M. Rockett, Arc Length, Area, and the Arcsine Function. *Mathematics Magazine*, 56(1983), no 2, 104-110.

[12] Peter L. Walker, Elliptic functions: A constructive approach, New York: Wiley, 1996.

Literature on Elliptic functions

[13] Appell, Paul; Lacour, Emile and Garnier, Rene. Principes de la Theorie des Fonctions Elliptiques et Applications, 2. ed., revue et augmentee. Paris: Gauthier-Villars, 1922. 503 p.

[14] Ayoub, Raymond. The lemniscate and Fagnano's contributions to elliptic integrals. *Arch. Hist. Exact Sci.* **29**(1984), no. 2, 131–149.

[15] Bobek, Karl. Einleitung in die Theorie der elliptischen Funktionen. Leipzig, Germany: Teubner, 1884. 274 p.

[16] Bowman, Frank. Introduction to Elliptic Functions, with Applications. New York: Dover, 1961. 115 p.

[17] Briot and Bouquet. Theorie des fonctions elliptiques 2eme edition. Paris: Gauthier-Villars, 1875. 700 p.

[18] Dixon, Andrew Cardew. The Elemetary Properties of the Elliptic Functions with Examples. London: Macmillan, 1894. 142 p.

[19] Du Val, Patrick. Elliptic Functions and Elliptic Curves. London: Cambridge University Press, 1973. 248 p.

[20] Durege, H. Theorie der Ellipischen Funktionen, in funfter Auflage neu Bearbeitet von Ludwig Maurer. Leipzig, Germany: Teubner, 1908. 436 p.

[21] Dutta, Mahadev. and Debnath, Lokenath. Elements of the Theory of Elliptic and Associated Functions with Applications. Calcutta: World Press, 1965. 290 p.

[22] Eagle, Albert. The Elliptic Functions as They Should Be: An Account, with Applications, of the Functions in a New Canonical Form. Cambridge, England: Galloway and Porter, 1958. 508 p.

[23] Enneper, Alfred. Elliptische Functionen: Theorie und Geschichte, 2. aufl. Halle a. S., Germany: Louis Nebert, 1890. 598 p.

[24] Euler, L. Opera Omnia, Vol. 20. Leipzig, Germany, 1912.

[25] Fricke, Robert. Die elliptischen Funktionen und ihre Anwendungen. Leipzig: B.G. Teubner, 1916.

[26] Graeser, Ernst. Einfuhrung in die Theorie der Elliptischen Funktionen und deren Anwendungen. Munich, Germany: Oldenbourg, 1950. 144 p.

[27] Greenhill, Alfred George. The Applications of Elliptic Functions. London: Macmillan, 1892. Reprinted New York: Dover, 1959. 357 p.

[28] Halphen, G.-H. Traite des fonctions elliptiques et leurs applications, premier partie. Paris: Gauthier-Villars.

[29] Halphen, G.-H. Traite des fonctions elliptiques et leurs applications, deuxieme partie: Applications a la mechanique, a la physique, a la geodesie, a la geometrie et au calcul integral. Paris: Gauthier-Villars, 1888. 659 p.

[30] Halphen, G.-H. Traite des fonctions elliptiques et leurs applications, troiseme partie: Fragments. Paris: Gauthier-Villars, 1891. 272 p.

[31] Hancock, Harris. Lectures on the Theory of Elliptic Functions. New York: Wiley, 1910. 498 p.

[32] Hermite, C. Oeuvres Mathematiques. Paris, 1905-1917.

[33] Jacobi, Carl Gustav Jakob. Fundamentia Nova Theoriae Functionum Ellipticarum. Konigsberg, Germany: Regiomonti, Sumtibus fratrum Borntraeger, 1829. Reprinted in Gesammelte Mathematische Werke, Vol. 1, pp. 497-538.

[34] Jordan, Camilie. Fonctions Elliptiques. 1981.

[35] King, Louis Vessot. On the Direct Numerical Calculation of Elliptic Functions and Integrals. Cambridge, England: Cambridge University Press, 1924. 42 p.

[36] Klein, Felix. Vorlesungen uber die Theorie der elliptischen Modulfunctionen, 2 vols. Leipzig, Germany: Teubner, 1890-92.

[37] Koenignsberger, Leo. Vorlesungen uber die Theorie der elliptischen Functionen, 2 theilen. Leipzig, Germany: Teubner, 1874. 431+219 p.

[38] Konig, R. and Kraft, M. Elliptische Funktionen. Berlin: de Gruyter, 1928. 259 p.

[39] Lang, Serge. Elliptic Functions, 2nd ed. New York: Springer-Verlag, 1987. 326 p.

[40] Laurent, H. Theorie elementaire des fonctions elliptiques. Paris: Gauthier-Villars, 1882. 182 p.

[41] Legendre, A.-M. Theorie des Fonctions Elliptiques. 1895.

[42] Murty, M. Ram (Ed.). Theta Functions. Providence, RI: Amer. Math. Soc., 1993. 174 p.

[43] Neville, Eric Harold. Jacobian Elliptic Functions, 2nd ed. Elsevier, 1971. 345 p. Out of print.

[44] Oberhettinger, Fritz and Magnus, Wilhelm. Anwendung der Elliptischen Funktionen in Physik und Technik. Berlin: Springer-Verlag, 1949.

[45] Prasolov, Viktor and Solovyev, Yuri. Elliptic Functions and Elliptic Integrals. Providence, RI: Amer. Math. Soc., 1997. 185 p.

[46] Slavutin, E. I., Euler's works on elliptic integrals. (Russian) History and methodology of the natural sciences, No. XIV: Mathematics (Russian), pp. 181–189. Izdat. Moskov. Univ., Moscow, 1973.

[47] Study, E. Spharische Trigonometrie, orthogonale Substitutionen, und elliptische Functionen: Eine Analytisch-Geometrische Untersuchung. Leipzig, Germany: Teubner, 1893. 231 p.

[48] Tannery, Jules and Molk, Jules. Elements de la Theorie des Fonctions Elliptiques, tomes I et II, 2nd ed. Tome I: Introduction, Calcul Differentiel (1ere partie). Tome II: Calcul Differentiel (2ere partie). Paris: Gauthier-Villars, 1893-1902. Reprinted Bronx, NY: Chelsea, 1972. 246+299 p.

[49] Tannery, Jules and Molk, Jules. Elements de la Theorie des Fonctions Elliptiques, tomes III et IV, 2nd ed. Tome III: Calcul Integral (1ere partie), Theorenes Generaux, Inversion. Tome IV: Calcul Integral, Inversion. Paris: Gauthier-Villars, 1893-1902. Reprinted Bronx, NY: Chelsea, 1972. 267+303 p.

[50] Tolke, Friedrich. Praktische Funktionenlehre, zweiter Band: Theta-Funktionen und spezielle Weierstra?sche Funktionen. Berlin: Springer-Verlag, 1966. 248 p.

[51] Tolke, Friedrich. Praktische Funktionenlehre, dritter Band: Jacobische elliptische Funktionen, Legendresche elliptische Normalintegrale und spezielle Weierstrassische Zeta- und Sigma Funktionen. Berlin: Springer-Verlag, 1967. 180 p.

[52] Tolke, Friedrich. Praktische Funktionenlehre, vierter Band: Elliptische Integralgruppen und Jacobische elliptische Funktionen im Komplexen. Berlin: Springer-Verlag, 1967. 191 p.

[53] Tolke, Friedrich. Praktische Funktionenlehre, funfter Band: Allgemeine Weierstrassische Funktionen und Ableitungen nach dem Parameter. Integrale der Theta-Funktionen und Bilinear-Entwicklungen. Berlin: Springer-Verlag, 1968. 158 p.

[54] Tolke, Friedrich. Praktische Funktionenlehre, sechster Band, erster Teil: Tafeln aus dem Gebiet der Theta-Funktionen und der elliptischen Funktionen mit 120 erlauternden Beispielen. Berlin: Springer-Verlag, 1969. 449 p.

[55] Tolke, Friedrich. Praktische Funktionenlehre, sechster Band, zweiter Teil: Tafeln aus dem Gebiet der Theta-Funktionen und der elliptischen Funktionen. Berlin: Springer-Verlag, 1970. 1047 p.

[56] Venkatachaliengar, K. Development of Elliptic Functions According to Ramanujan. Technical Report, 2. Madurai Kamaraj University, Department of Mathematics, Madurai, India, n.d. 147 p.

[57] Weber, Heinrich. Elliptische Funktionen und algebraische Zahlen. Brunswick, Germany, 1891.

[58] Whittaker, Edmund Taylor and Watson, G.N. Ch. 20-22 in A Course of Modern Analysis, 4th ed. Cambridge, England: University Press, 1943.