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PROV I MATEMATIK

Partiella differentialekvationer D
1998 03 16, kl. 08.00–14.00

8tlen

You are allowed to use: Beta or corresponding list of formulas.
Grades: 18 points give the grade *Godkänd*; 28 the grade *Väl godkänd.*
A good lecture gives 7 extra points; an excellent lecture 9 extra points.

1. Find a solution to the quasilinear differential equation

$$(x+u+1)\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = x+u-1$$

which is zero when $y = 1$. [6 points]

2. Consider the nonlinear equation

$$\frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial x}\right)^2.$$

(a) Prove that there is no real solution in any open set which contains the straight line $y = x$ and which satisfies $u(x,x) = x^2/2$. (b) Prove that, by way of contrast, the equation does have a solution which is defined for $y < x + 3$ and which satisfies $u(x,x) = e^x$, $x \in \mathbf{R}$. (c) Prove that there is no real solution in all of $\mathbf{R}^2$ which satisfies $u(x,x) = e^x$, $x \in \mathbf{R}$. *Hint:* Choose $x, y \in \mathbf{R}$ such that the equation $x + 2py = (1+2p)\log(p+p^2)$ has more than one solution p. [7 points]

3. Study the equation

$$\frac{3}{4}\frac{\partial^2 u}{\partial x^2} - 2y\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\frac{\partial u}{\partial x} = 0.$$

(a) Where is the equation hyperbolic? (b) Determine the characteristic curves. (c) Transform the equation to canonical form where this is possible. (d) Determine its general solution in the domain where it is hyperbolic. [7 points]

4. Solve the mixed problem

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \text{ for } x > 0,\ t > 0;$$

$$\frac{\partial u}{\partial x}(0,t) = 0,\ t > 0; \qquad u(x,0) = 0,\ x > 0; \qquad \frac{\partial u}{\partial t}(x,0) = 2cxe^{-x},\ x > 0.$$

[7 points]

PTO. Fortsättning på andra sidan! There is more to be done on the other side!

5. Let us consider a modified wave operator

$$Pu = \frac{\partial^2 u}{\partial t^2}(x,t) - c^2 \Delta_x u(x,t) + a(x,t)\frac{\partial u}{\partial t} + b(x,t)u(x,t)$$

för $t > 0,\ x \in \Omega \subset \mathbf{R}^n$. Here c is a constant and $a(x,t) \geqslant 0$, $b(x,t) \geqslant 0$, while $\frac{\partial b}{\partial t}(x,t) \leqslant 0$. Prove that the problem

$$\begin{aligned} &Pu = f \text{ when } t > 0,\ x \in \Omega, \\ &u = g_0 \text{ and } \frac{\partial u}{\partial t} = g_1 \text{ when } t = 0,\ x \in \Omega, \\ &u = h \text{ when } t \geqslant 0,\ x \in \partial\Omega, \end{aligned}$$

has at most one solution for given continuous functions f, g_0, g_1, h. *Hint:* Study, for a suitable u, the functional

$$E(t) = \frac{1}{2}\int_\Omega \left(u_t^2 + c^2|\nabla_x u|^2 + bu^2\right) dx_1 \cdots dx_n, \qquad t \geqslant 0.$$

[7 points]

6. Prove that if P is the differential operator

$$Pu = u - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^3 u}{\partial x_2^3},$$

then there is a fundamental solution which is a continuous function in the whole plane, viz.

$$E(x) = (2\pi)^{-2}\int_{\mathbf{R}^2} \frac{e^{ix_1\xi_1 + ix_2\xi_2}}{1 + \xi_1^2 + i\xi_2^3} d\xi_1 d\xi_2.$$

To prove this you have to investigate the convergence of the integral. (That E is a fundamental solution means that $P(D)E = \delta$, which shall be interpreted as

$$\int_{\mathbf{R}^2} E(x)P(-D)\varphi(x)dx_1dx_2 = \varphi(0) \text{ for all test functions } \varphi.)$$

[6 points]