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Matematiska institutionen
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PROV I MATEMATIK

Partiella differentialekvationer D
1998 08 29, kl. 08.00–14.00

8t3pde

You are allowed to use: Beta or corresponding list of formulas.
Grades: 18 points give the grade *Godkänd*; 28 the grade *Väl godkänd.*
A good lecture gives 7 extra points; an excellent lecture 9 extra points.

1. Find a solution to the quasilinear differential equation

$$2u(2u+y)\frac{\partial u}{\partial x} + (2u+y)\frac{\partial u}{\partial y} = u$$

which is defined in a halfplane $\{(x,y) \in \mathbf{R}^2;\, x > a\}$ for some a with $a < 1$ and which satisfies $u(x,1)^2 + u(x,1) = x$ for $x > a$ as well as $u(2,1) = 1$. Try to determine the largest halfplane, i.e., try to find an a which is as small as possible. [6 points]

2. Consider the nonlinear equation

$$\left(\frac{\partial u}{\partial t}\right)^2 = \frac{\partial u}{\partial x}.$$

(a) Prove that there is no real-valued solution in any open set which contains the straight line $t = 0$ and which satisfies $u(x,0) = x^2$, $x \in \mathbf{R}$. (b) Prove (for instance using the method of envelopes) that the equation does have a solution which is defined in some open set containing the line $t = 0$ and which satisfies $u(x,0) = e^x$, $x \in \mathbf{R}$. (c) Prove that the solution found in (b) satisfies $u(x,t) \geqslant e^x + te^{x/2}$ near every point on the axis $t = 0$.

[7 points]

3. Study the equation

$$2\frac{\partial^2 u}{\partial x^2} - 2y^3\frac{\partial^2 u}{\partial y^2} - 3y^2\frac{\partial u}{\partial y} = 0.$$

(a) Where is the equation hyperbolic? (b) Determine the characteristic curves. (c) Transform the equation to canonical form where this is possible. (d) Determine its general solution in the domain where it is hyperbolic. [7 points]

PTO. Fortsättning på andra sidan! There is more to be done on the other side!

4. Solve the mixed problem

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \text{ for } x > 0, \ t > 0;$$

$$u(0,t) = 1, \ t > 0; \qquad u(x,0) = e^{-x^2}, \ x > 0; \qquad \frac{\partial u}{\partial t}(x,0) = 0, \ x > 0.$$

[7 points]

5. Let $\Omega = \{x \in \mathbf{R}^3; \ |x| < \pi\}$ be the open ball in $\mathbf{R}^3$ with radius π and center at the origin (Euclidean norm). Prove that the Dirichlet problem

$$\Delta u + u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

can have a solution u only if

$$\int_\Omega f(x) \frac{\sin|x|}{|x|} dx = 0.$$

(So this Dirichlet problem is very different from that for the Laplace operator.) [7 points]

6. Let P be the differential operator

$$Pu = 2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y}.$$

Prove that a constant c times the characteristic function of the set

$$A = \{(x,y) \in \mathbf{R}^2; \ x > \max(0, 2y)\}$$

is a fundamental solution for P. Determine c. (That E is a fundamental solution means that $P(D)E = \delta$, which shall be interpreted as

$$\int_{\mathbf{R}^2} E(x,y) P(-D) \varphi(x,y) \, dxdy = \varphi(0,0) \text{ for all test functions } \varphi.)$$

[6 points]