PDE: Tenta 2001-03-12

Allowed material: Calculators, books and notes.

1.) Find the integral curve $\gamma(t) = (x(t), y(t), z(t))$ of the vector field $X(x,y,z) = (yz, z, 2x - y^2)$ with $\gamma(0) = (1,1,1)$.

2.) Solve the Cauchy problem

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u^2 \ , \ u(x,0) = x \ .$$

3.) Solve the Cauchy problem

$$(\frac{\partial u}{\partial x})^3 + \frac{\partial u}{\partial y} = u \ , \ u(x,0) = x \ .$$

4.) Solve in $\mathbf{R}^2_{\geq 0}$ (the first quadrant) the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \ , u(x,0) = x^2, u(0,t) = 0, \frac{\partial u}{\partial t}(x,0) = 0 \ .$$

Hint: Think about even and odd functions!

5.) Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ be a continuous function, $f \geq 0$ and $\int_{\mathbf{R}^n} f(x)dx = 1$. Show that for any test function $\varphi \in \mathcal{D}(\mathbf{R}^n)$ one has

$$\lim_{r\to\infty} \int_{\mathbf{R}^n} r^n f(rx)\varphi(x)dx = \varphi(0) \ .$$

6.) Let $u \in C^2(\overline{\Omega})$ be the temperature distribution in an insulated body (realized as the closure $\overline{\Omega}$ of a bounded open set $\Omega \subset \mathbf{R}^3$ with smooth boundary). That means that $\frac{\partial u}{\partial n}|_{\partial\Omega} = 0$. Show that the total energy

$$E(t) := \int_\Omega u(x,t)dx$$

is constant.

7.) Consider the differential operator

$$Lu := \Delta u + a(x) \cdot \nabla u + b(x)u$$

on a bounded open set $\Omega \subset \mathbf{R}^n$. Here $a : \Omega \longrightarrow \mathbf{R}^n, b : \Omega \longrightarrow \mathbf{R}$ are continuous functions and $b(x) < 0$, $\forall x \in \Omega$. Show that for a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ the assumption $Lu = 0, u|_{\partial\Omega} = 0$ already implies $u = 0$. **Hint:** Look at the maximum and minimum of u at $\overline{\Omega}$!

8.) Find a function $u \in C^2(\mathbf{R}^3)$, such that

$$\Delta u = \max\{1 - ||x||, 0\} \ , \ u(1,2,1) = 0 \ .$$

LYCKA TILL!