PDE: Tenta 2001-08-22

Allowed material: Calculators, books and notes.

1.) Determine the characteristic curves for the equation

$$\frac{\partial^2 u}{\partial x^2} - 9x^4 \frac{\partial^2 u}{\partial y^2} - 6xu = 0$$

in the domain $\Omega := \{(x,y); x > 0\}$, i.e. the right half plane.

2.) Solve the Cauchy problem

$$(x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y})u = y^2 - x^2,\ u(s,s) = 2s\ .$$

Hint: The equation can be written as an inhomogeneous linear equation for a suitable function!

3.) Solve the Cauchy problem

$$\frac{\partial u}{\partial y} = (\frac{\partial u}{\partial x})^4,\ u(0,y) = y\ .$$

4.) Show that a (real valued) harmonic function $u(x,y)$ on the unit disk $E := \{z = x+iy; x^2 + y^2 < 1\}$ is the real part of a holomorphic function on E. Derive from that a necessary and sufficient condition on a function $h : \,]-1,1[\longrightarrow \mathbf{R}$ in order that there exists a function $u : E \longrightarrow \mathbf{R}$ satisfying $\Delta u = 0; u(x,0) = h(x)$.

5.) Show that a distribution $u \in \mathcal{D}'(\mathbf{R})$ satisfying $u' = 0$ is constant, i.e. there is a real number $c \in \mathbf{R}$, such that $u(\varphi) = \int_{\mathbf{R}} c\varphi(x)dx$ for all test functions $\varphi \in \mathcal{D}(\mathbf{R})$. Hint: When does a test function have a primitive which itself is a test function?

6.) Show that the function $f(x,y) = \frac{1}{2\pi}\ln(r), r = \sqrt{x^2+y^2}$, is locally integrable on the whole plane $\mathbf{R}^2$ and compute the distribution $\Delta f \in \mathcal{D}'(\mathbf{R}^2)$.

7.) Solve the differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}$$

with the initial conditions $u(x,0) = x$, $\frac{\partial u}{\partial t}(x,0) = 0$. Hint: Apply the same method as for the wave equation!

8.) Assume that the functions $a, b : \mathbf{R}^2 \longrightarrow \mathbf{R}$ satisfy $a(x,y)x + b(x,y)y > 0$ for $(x,y) \in S^1$. Show that any solution of the PDE

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = -u$$

which is defined in a neighbourhood of the closed unit disk $\overline{E} = \{(x,y); x^2+y^2 \leq 1\}$ vanishes identically on $\overline{E}$. Hint: The function u takes its maximum and minimum on $\overline{E}$, either in its interior E or on the boundary $\partial E = S^1$.

LYCKA TILL!