PDE: Tenta 2002-03-11

Allowed material: Text book, formulary and own notes.

1.) Compute the characteristic curves of the differential operator

$$L = xy(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}) + (x^2 - y^2)\frac{\partial^2}{\partial x \partial y}$$

Hint: Construct two different vectorfields X, Y, such that $T_aM = X(a)^\perp$ or $T_aM = Y(a)^\perp$, whenever $M \subset \mathbf{R}^2$ is a characteristic curve of L passing through $a \in \mathbf{R}^2 \setminus \{0\}$. Then try to find potentials $\varphi, \psi : \mathbf{R}^2 \longrightarrow \mathbf{R}$ for those vector fields, i.e. $X = \nabla\varphi, Y = \nabla\psi$.

2.) Solve the Cauchy problem

$$\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \ , \ u(0, y) = f(y) \ .$$

3.) Solve the Cauchy problem

$$(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 - 4u = 0 \ , \ u(x, 0) = \frac{x^2}{2} \ .$$

4.) Solve in $\mathbf{R}^2_{\geq 0}$ (the first quadrant) the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \ , u(x, 0) = x^4, u(0, t) = 0, \frac{\partial u}{\partial t}(x, 0) = \sin(x) \ .$$

Hint: Reduce by reflection to a similar problem in the upper half space $\mathbf{R} \times \mathbf{R}_{\geq 0}$!

5.) Determine all solutions $u \in \mathcal{D}'(\mathbf{R})$ of the equation $u' = 0$! For a given distribution $u \in \mathcal{D}'(\mathbf{R})$ find a primitive $v \in \mathcal{D}'(\mathbf{R})$, i.e. $v' = u$.

6.) Solve the Cauchy problem

$$x \cdot \nabla u = u, \quad u|_{S^{n-1}} = f$$

on $\mathbf{R}^n \setminus \{0\}$! When is the corresponding Cauchy problem on the entire space $\mathbf{R}^n$ solvable (with a strong (non-weak) solution)?

7.) Define the function $\psi : \mathbf{R}^2 \setminus \{0\} \longrightarrow \mathbf{R}$ by

$$\psi(x, t) = \begin{cases} \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} & t > 0 \\ 0 & t \leq 0 \end{cases} \ .$$

Show that the regular distribution ψ^* is a fundamental solution of the heat operator $L := \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$. (You may use without proof: $\int_{\mathbf{R}} e^{-x^2} dx = \sqrt{\pi}$.)

8.) Find a function $u \in C^2(\mathbf{R}^4 \setminus \{0\})$, such that

$$\Delta u(x) = ||x||^5 \ , \ u|_{S^3} = 0 \ .$$

LYCKA TILL!