PDE: Tenta 2002-06-04

Allowed material: Text book, formulary and own notes.

1.) Find the characteristic curves of the differential operator

$$Lu := x\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \ .$$

2.) Solve the Cauchy problem

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u^2 \ , \ u(0,y) = y$$

near the origin.

3.) Solve the following Cauchy problem:

$$(\frac{\partial u}{\partial x})^3 + \frac{\partial u}{\partial y} = u; \quad u(x,0) = x \ .$$

4.) Show that there is no locally integrable function f on $\mathbf{R}$ with $\delta = f^*$! Instead give a sequence of functions $f_n : \mathbf{R} \longrightarrow \mathbf{R}$, such that $\delta(\varphi) = \lim_{n\to\infty} f_n^*(\varphi)$ for any test function $\varphi \in \mathcal{D}(\mathbf{R})$.

5.) Show that

$$u(\varphi) := \lim_{\varepsilon\to 0}(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty})\frac{\varphi(x)}{x}dx$$

defines a distribution $u \in \mathcal{D}'(\mathbf{R})$ and that u is the derivative of the regular distribution f^* with the function $f(x) := |\ln(x)|$. Hint: $\varphi = (\varphi - \varphi(0)) + \varphi(0)$.

6.) Show that the function $f(x,y,z) = \frac{1}{r}, r := \sqrt{x^2+y^2+z^2}$, is locally integrable on $\mathbf{R}^3$ and compute the (non regular) distribution Δf^*.

7.) Let $\Omega \subset \mathbf{R}^n$ be a bounded domain. Assume that $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ vanishes on $\partial\Omega$ and satisfies

$$\Delta u = u^3 - u \ .$$

Show that $-1 \leq u \leq 1$. Hint: Consider the maximum and minimum of u.

8.) Find a function $u(x,t); x \geq 0, t \geq 0$ satisfying

$$u_{xx} = u_{tt} \ , u(x,0) = 0 = u_t(x,0), u(0,t) = h(t) \ ,$$

where $h \in C^2(\mathbf{R}_{\geq 0}), h(0) = 0$.

Results are available in the beginning of august.

LYCKA TILL!