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Some important examples

- Inviscid Burger's (Hopf) equation

$$u'_t + uu'_x = 0$$

- Scalar conservation law

$$u'_t + \operatorname{div} \mathbf{F}(u) = 0$$

- Laplace Equation

$$\Delta u \equiv u''_{xx} + u''_{yy} = 0$$

- Poisson's equation

$$\Delta u \equiv u''_{xx} + u''_{yy} = f(x, y)$$

- Heat (or diffusion) equation

$$u'_t - \Delta u = 0$$

- Wave equation

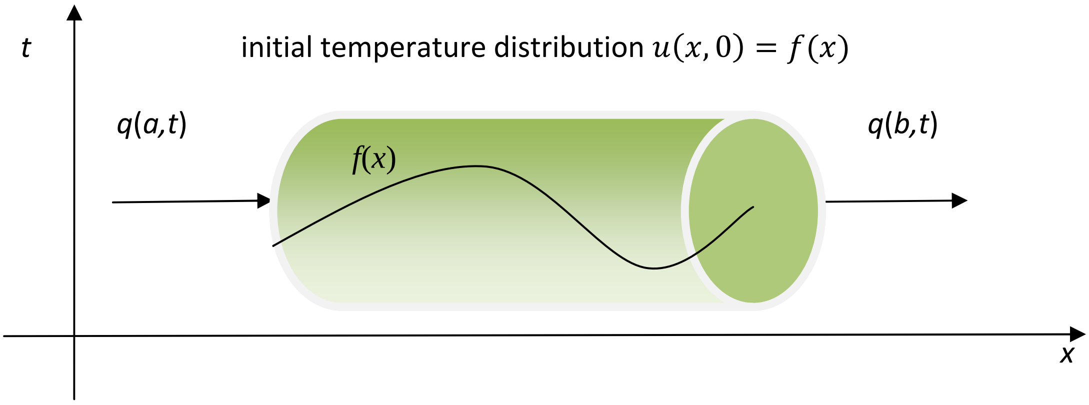
$$u''_{tt} - \Delta u = 0$$

- Minimal surface equation

$$\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0$$

Derivation of the heat equation

We consider the flow of heat along a metal *insolated* rod



Energy of an arbitrary piece of rod from a to b is

$$E = \int_a^b A \cdot \rho\, c \cdot u(x,t)\, dx$$

- $u = u(x,t)$ is temperature at time *t* at a given point x,
- A is the cross sectional area of the rod,
- c is the specific heat capacity of the rod

The wave heat flow:

$$R = A\big(q(a,t) - q(b,t)\big) = -A \int_a^b \frac{\partial}{\partial x} q(x,t)\, dx$$

Conservation of energy (in terms of power = time-derivative of energy):

$$R = \frac{\partial E}{\partial t}$$

implies the *integral form* of the heat equation:

$$\int_a^b \left(\rho\, c \cdot \frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} q(x,t) \right) dx = 0.$$

By virtue of arbitrariness of a and b we get

$$\rho\, c \cdot \frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} q(x,t) = 0.$$

Finally, by using the Fourier law $q(x,t) = -\lambda \frac{\partial}{\partial x} q(x,t)$ we arrive at (the differential form of) the heat equation:

$$\frac{\partial u}{\partial t} - C\, \frac{\partial^2 u}{\partial x^2} = 0.$$

Classification of the 1st order PDE's

Standard notation:

For $u = u(x,y)$ one denotes the first derivatives by

$p = u'_x$

$q = u'_y$

- The most general 1st order PDE

$$F(x,y,u,p,q) = 0$$

- A **linear** equation

$$a(x,y)u'_x + b(x,y)u'_y = c(x,y)$$

- A **homogeneous** (linear) equation

$$a(x,y)u'_x + b(x,y)u'_y = 0$$

Generalizations of the linear case:

- A **semilinear** equation

$$a(x,y)u'_x + b(x,y)u'_y = c(x,y,u)$$

- A **quasilinear** equation

$$a(x,y,u)u'_x + b(x,y,u)u'_y = c(x,y,u)$$

Fully non-linear equation:

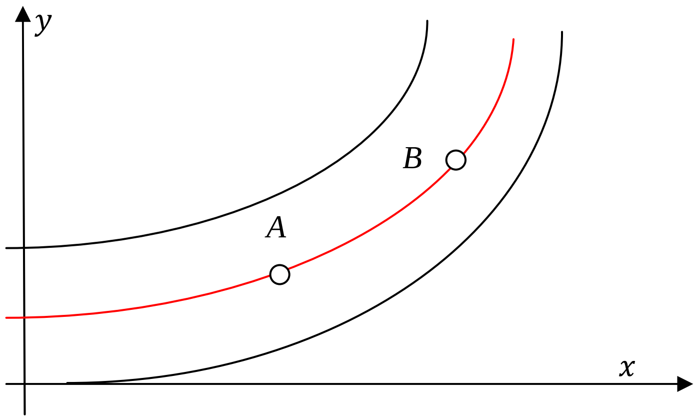
$$F(x,y,u,p,q) = 0$$

with *F* chosen arbitrarily; then additionally required that

$$F_p'^2 + F_q'^2 \neq 0$$

1. Characteristics for a homogeneous linear equation

$$a(x,y)u'_x + b(x,y)u'_y = 0$$



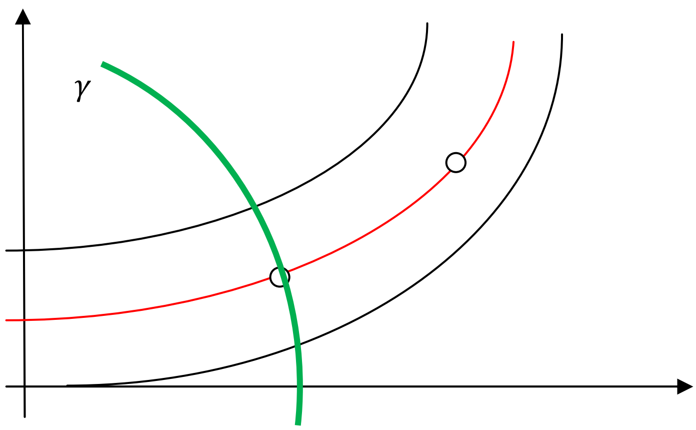
Characteristic equations:

$$\frac{dx}{dt} = a(x,y), \qquad \frac{dy}{dt} = b(x,y)$$

this yields

$$\frac{du(x(t),y(t))}{dt} = 0$$

hence $u(x,y) = \text{const}$ along each characteristic curve. In particular $u(A) = u(B)$ on the picture above and one can determine solution (uniquely) if one knows the values of the solution at some points. For instance, if one knows the values along a curve γ which is transversal to characteristic curves:



2. The method of characteristics for general quasilinear equation

$$a(x,y,u)u'_x + b(x,y,u)u'_y = c(x,y,u)$$

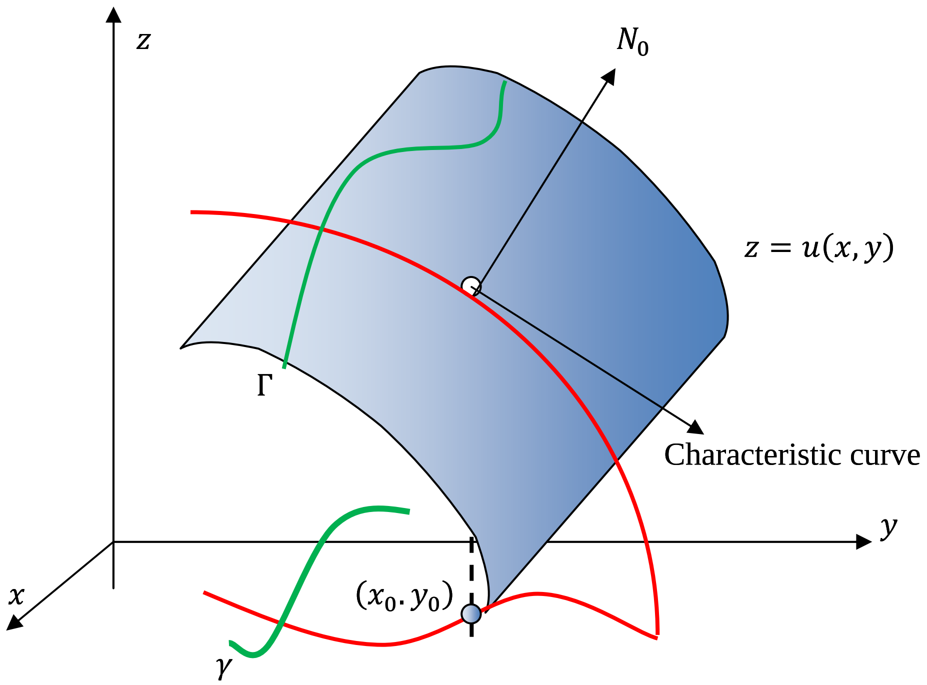
If we introduce a vector field $V = (V_1, V_2, V_3)$ with coordinates

$$V_1 = a(x,y,u), \qquad V_2 = b(x,y,u), \qquad V_3 = c(x,y,u)$$

then V is orthogonal to the normal vector

$$N_0 = (-u'_x(x_0.y_0), -u'_y(x_0.y_0), 1)$$

at the point $(x_0, y_0, u(x_0, y_0))$ on the graph of a solution $z = u(x,y)$:



- $z = u(x,y)$ are *integral surfaces* of the vector field V
- a characteristic curve (in red)
- a Cauchy data Γ (in green): a curve in $\mathbb{R}^3$ transversal to the vector field V

Characteristic equations:

$$\frac{dx}{dt} = a(x,y,u), \qquad \frac{dy}{dt} = b(x,y,u), \qquad \frac{dz}{dt} = c(x,y,u)$$

The Cauchy problem: given a curve Γ in $\mathbb{R}^3$, find a solution u of the 1st order equation whose graph contains Γ:

$$u|_\gamma = h(x,y)\ .$$

Fully non-linear equation:

$$F(x, y, z, p, q) = 0$$

Where as usual $z = u(x, y)$, $p = u'_x$ and $q = u'_y$, and

$$F_p'^2 + F_q'^2 \neq 0$$

The latter implies that locally either p or q can be found as a function of the remaining variables.

We have seen in the case F is linear with respect to p, q that the normal vector field to the graph $z = u(x, y)$ is orthogonal to the vector field $V = (-F'_p, -F'_q, 1)$. In that case the Cauchy problem $u|_\gamma = u_0(x, y)$ is well-posed if the curve γ is transversal to all characteristics it meets.

In order to adjust the characteristics method one needs to "linearize" the initial non-linear equation. An idea is to show that the first derivatives $p = u'_x$ and $q = u'_y$ satisfy *quasilinear* equations.

Namely, differentiating w.r.t. x and y yields two quasilinear equations for p, q:

$$F'_x + F'_z u'_x + F'_p p'_x + F'_q q'_x = 0 \qquad (1)$$

and

$$F'_y + F'_z u'_y + F'_p p'_y + F'_q q'_y = 0 \qquad (2)$$

Indeed, we have for the mixed partial derivatives: $p'_y = u''_{xy} = u''_{yx} = q'_x$, hence eq. (1) and (2) take the quasilinear form

$$F'_p p'_x + F'_q p'_y = -F'_x - F'_z p$$

$$F'_p q'_x + F'_q q'_y = -F'_y - F'_z q$$

Applying the characteristic equations to this system we get

$$\frac{dx}{dt} = F'_p, \qquad \frac{dy}{dt} = F'_q, \qquad \frac{dp}{dt} = -F'_x - F'_z p, \qquad \frac{dq}{dt} = -F'_y - F'_z q.$$

We need only one equation for z. One can get by differentiating $z = u(x, y)$ w.r.t. t subject to the first previous equations:

$$\frac{dz}{dt} = \frac{d}{dt}u(x, y) = u'_x \frac{dx}{dt} + u'_y \frac{dy}{dt} = p \cdot F'_p + q \cdot F'_q$$

Thus we have arrived at the following system

$$\frac{dx}{dt} = F'_p,$$

$$\frac{dy}{dt} = F'_q,$$

$$\frac{dz}{dt} = p \cdot F'_p + q \cdot F'_q$$

$$\frac{dp}{dt} = -F'_x - F'_z p,$$

$$\frac{dq}{dt} = -F'_y - F'_z q.$$

This system determines a family of integral curves in $\mathbb{R}^5 = \mathbb{R}^2_{xy} \times \mathbb{R}^2_{pq} \times \mathbb{R}^1_z$ and it is called the characteristic equations for the non-linear equation $F = 0$.

In **general**, in the n-dimensional case one has a system of similar equations in $\mathbb{R}^{2n+1}$. In fact, let we have a 1st order non-linear equation

$$F(x, u, Du) = 0$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $n \geq 2$, and $Du = \left(u_{x_1}, \dots, u_{x_n}\right) = (p_1, \dots, p_n)$ is the gradient of $u = u(x_1, \dots, x_n)$. Then the modified system for characteristics is

$$\frac{dx_k}{dt} = F'_{p_k},$$

$$\frac{dp_k}{dt} = -F'_{x_k} - F'_z p_k,$$

for $k = 1, \dots, n$, and

$$\frac{dz}{dt} = DF \cdot p = \sum_{k=1}^{n} F'_{p_k} p_k.$$

Return to $n = 2$. We must complete our Cauchy conditions because we have now 5 ODE's but only 3 initial Cauchy conditions. Since now we are in

$$\mathbb{R}^5 = \mathbb{R}^2_{xy} \times \mathbb{R}^2_{pq} \times \mathbb{R}^1_z$$

it is clear that we need only the Cauchy data p_0 and q_0.

- We recall that the Cauchy condition can be written as a parameterized curve:

  $$\Gamma: \quad x = x_0(s), \;\; y = y_0(s), \;\; z = z_0(s)$$

  Substituting this into $F(x, y, z, p, q) = 0$ yields

  $$F(x_0, y_0, z_0, p_0, q_0) = 0 \qquad \text{(IC-1)}$$

- Another relation is found by differentiating the original initial condition (IC) with respect to the inner parameter s:

$$\frac{d}{ds}z_0(s) = \frac{d}{ds}u\big(x_0(s), y_0(s)\big) = u'_x\big(x_0(s), y_0(s)\big) \cdot \frac{dx_0}{ds} + u'_y\big(x_0(s), y_0(s)\big) \cdot \frac{dy_0}{ds}$$

  This yields the so-called *strip condition*:

$$\frac{d}{ds}z_0(s) = p_0(s) \cdot \frac{dx_0}{ds} + q_0(s) \cdot \frac{dy_0}{ds} \qquad \text{(IC-2)}$$

These equations (IC-1) - (IC-2) provide two additional initial data, for p_0 and q_0.

In fact p_0 and q_0 need not to be uniquely defined and need not even exist. However, once p_0 and q_0 do exist, one can determine an integral surface

$$x = x(s,t), \;\; y = y(s,t), \;\; z = z(s,t)$$

which gives a parametric form for the solution of the Cauchy problem for the non-linear equation $F = 0$.

Remark: Our notation p_0 and q_0 here correspond to φ and ψ given in MacOwen, p. 34-35.

Method of envelopes

In general, for the 1st order non-linear equation

$$F(x, u, Du) = 0 \qquad (*)$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $n \geq 2$, we set vector-notation

$$D_pF = (F_{p_1}, F_{p_2}, \dots, F_{p_n})$$

$$D_xF = (F_{x_1}, F_{x_2}, \dots, F_{x_n})$$

(We assume that F is smooth, at least of class C^2 in some domain in $\mathbb{R}^{2n+1}$).

We are concerned with finding solutions u of (*) in some open set $U \subset \mathbb{R}^n$, subject to the Cauchy condition

$$u = h \qquad \text{on } \Gamma,$$

where Γ is a subset of the boundary ∂U.

Suppose that we have found a parametric family of general solutions, say $u = u(x, a)$. Then we write also

$$(D_a u,\ D_{xa}^2) := \begin{pmatrix} u_{a_1} & u_{x_1 a_1} & \cdots & u_{x_n a_1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{a_n} & u_{x_1 a_n} & \cdots & u_{x_n a_n} \end{pmatrix}$$

for the composed Jacobian of size $n \times (n + 1)$.

Definition: A function $u = u(x_1, \dots, x_n)$ of class C^2 is called a *complete integral* in $U \times A$ provided

(i) $u(x, a)$ solves (*) for each $a \in A$

and

(ii) rank $(D_a u,\ D_{xa}^2) = n$, $(x, a) \in U \times A$.

In other words, $u(x, a)$ depends on all the n independent parameters $a_1, \dots, a_n$.

Example 1. *Clairaut's equation* (in honor of Alexis C. de Clairault, 1713-1765)

$$x \cdot Du + f(Du) = u$$

For instance, if $n = 2$ one has

$$xu'_x + yu'_y + f\left(u'_x, u'_y\right) = u$$

Then a complete integral is

$$u(x, a) = a \cdot x + f(a)$$

Example 2. The eikonal equation from geometric optic is

$$|Du|^2 = u_{x_1}^2 + u_{x_2}^2 + \dots + u_{x_n}^2 = 1$$

A complete integral is an *affine function*

$$u(x; a, b) = a \cdot x + f(b),$$

where $|a| = 1$, $b \in \mathbb{R}$.

Theorem 1. *Let $u(x;a)$ be a complete integral for $F=0$. Consider the vector equation*

$$D_a u(x;a) = 0 \qquad\qquad (**)$$

*Suppose we can solve it for a as a smooth function of x: $a=\varphi(x)$. Then the **envelope function** $v(x)=u(x;\ \varphi(x))$ solves also the original equation $F=0$.*

Remark: The method also works if one replaces one parameter, say a_n by a function of the remaining parameters, and substitute it into $u(x;a)$. This yields in general a wide choice of envelope solutions.

Idea of the proof: We have

$$v'_{x_k}(x) = \frac{\partial}{\partial x_k} u\big(x;\ \varphi(x)\big) = u'_{x_k}\big(x;\ \varphi(x)\big) + \sum_{i=1}^{n} u'_{a_i}\big(x;\ \varphi(x)\big)\cdot\frac{\partial \varphi_i}{\partial x_k}$$

where $u'_{a_i}=0$ for $a=\varphi(x)$ by virtue of our assumption (**). Hence

$$v'_{x_k}(x) = \frac{\partial}{\partial x_k} u\big(x;\ \varphi(x)\big), \qquad k=1,\dots,n$$

and it easily follows that the envelope function satisfies also $F(x,v,Dv)=0$. ■

How to apply?

We return again to $n=2$. Then a complete integral is denoted by $u(x,y;a,b)$ and it depends on independent parameters a and b. The above rank-condition is equivalent to saying that mapping

$$(a,b)\longrightarrow(u,u'_x,u'_y)$$

has rank 2 at each fixed x and y, that is the matrix

$$\begin{pmatrix} u'_a & u''_{ax} & u''_{ya} \\ u'_b & u''_{bx} & u''_{yb} \end{pmatrix}$$

has maximal rank.

In practice one usually uses a one parametric envelope solution which can be found by substituting some auxiliary function $b=B(a)$ or $a=A(b)$ in $u(x,y;a,b)$. We demonstrate this below.

Example 5. Consider $u'_x={u'_y}^2$ subject to initial condition $u(0,y)=\frac{y^2}{2}$.

Solution by the envelope method. An idea is to find solutions in the class of the $v=a+bx+cy+dxy$. The straightforward computation yields $d=0$, $b=c^2$, while a can be chosen arbitrarily. This gives after changing notation

$$v=a+b^2x+by$$

We see that the our Jacobian matrix has rank 2 (the first two columns):

$$\begin{pmatrix} u'_a & u''_{ax} & u''_{ya} \\ u'_b & u''_{bx} & u''_{yb} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2bx + y & 2b & 1 \end{pmatrix}$$

Hence now we are in position of Theorem 1.

(i) Set $a = kb^2$, where the constant k will be chosen later. We have

$$v = kb^2 + b^2x + by$$

and the envelope equation is

$$0 = \frac{\partial}{\partial b} v = 2kb + 2bx + y,$$

hence $b = -\frac{y}{2x+2k}$

(ii) Substituting this into v we find

$$v(x, y; a, b) = -\frac{y^2}{4(x+k)}$$

(iii) Finally applying our Cauchy condition we find $k = -\frac{1}{2}$. Hence the desired solution is

$$u(x, y) = \frac{y^2}{2 - 4x}$$

Question: Why $a = kb^2$? Check that the above argument breaks down for $a = kb$

Example 6. Consider

$$u'_x u'_y = u$$

Analys:

- $u = xy + ax + by + ab$ is a complete integral
- $u'_a = x + b, u'_b = y + a,$ hence we find $a = -y$ and $b = -x.$ This is the function φ in the Theorem.
- substituting φ into u yields: $u = xy + ax + by + ab = 0$
  which provides us another, trivial, solution.

Another choice is $b = a$. Then we get

$$u = xy + ax + ay + a^2$$

and $0 = u'_a = x + y + 2a$, hence $a = -\frac{x+y}{2}$.

Substituting this into u yields $u = -\frac{(x-y)^2}{4}$.