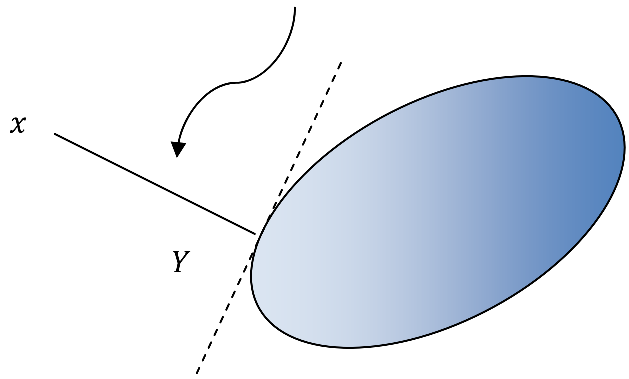
Eikonal equation

Let D be a compact (= closed and bounded) convex set in $\mathbb{R}^n$ and define the following distance function:

$$f(x) = \min_{y \in D} \|x - y\| = \|x - Y\|$$



This minimum is attained at some point $Y(x)$ which is uniquely determined by x, and it defines the map (the "convex projection")

$$x \to Y = Y(x)$$

From the triangle inequality we find

$$Y\big(x + t(Y(x) - x)\big) = Y(x)$$

That is all points on the ray $tx + Y(1 - t)$, $t \geq 0$, have the same convex projection x.

Another corollary of this property is that $f(x)$ is differentiable and the gradient of this function has the unit length:

$$\|\nabla f(x)\| = 1$$

and we have also the following boundary condition

$$f(x) = 0, \qquad x \in \partial D.$$

The latter identity is the simplest example of the *eikonal* equation. This equation with arbitrary R.H.S. arises in geometrical optics and describes the phase fronts of waves in (non)homogeneous media. The above minimum expresses the extremal property of light: it runs the minimal between two given objects.

In general, for general sets (not necessarily convex and on a general Riemann manifold) one should interpret solutions to the eikonal equation appropriately, for example, to allow Lipschitz functions, not necessarily differentiable everywhere.

The inviscid Burgers' equation

$$u\,u'_x + u'_y = 0$$

or equivalently

$$\left(\frac{u^2}{2}\right)'_x + u'_y = 0$$

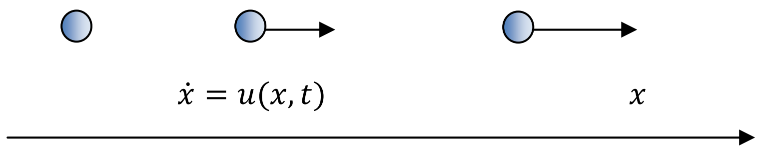
is an example of a conservation law in the general form

$$(G(u))'_x + u'_y = 0$$

Remark. This equation is a regular part of the **viscid Burgers' equation**

$$u\,u'_x + u'_y = \varepsilon\, u''_{xx}\,.$$

Mechanical interpretation. 1D stream of particles is in motion, each particle having constant velocity; a velocity field is given by $u(x,t)$, where t denotes time.



Hence, we are interested ho this model behaves for non-negative time: $t \geq 0$.

If we follow an individual particle, we get a function $x = x(t)$ for which $u(x(t),t)$ remains constant:

$$0 = \frac{d}{dt}\big(u(x(t),t)\big) = u'_x(x(t),t)\cdot\frac{dx(t)}{dt} + u'_y(x(t),t)\cdot\frac{dt}{dt} = u\,u'_x + u'_y$$

Then it is natural to assume that the initial velocity is given (to pose the Cauchy problem):

$$u(x,0) = h(x).$$

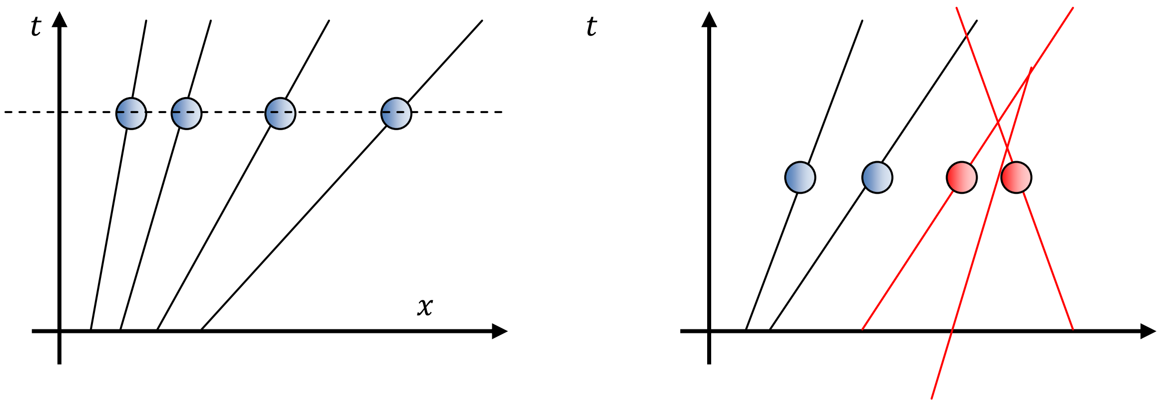
- Characteristics lines are straight lines:

$$x = h(s)t + s, \qquad y = t, \qquad z = h(s) \qquad (**)$$

- The general solution is $u = h(x - uy)$ – defines our solution *implicitly*
- The characteristics can cross; moreover, if the initial condition is differentiable then the "*breaking*" *time* can be found explicitly:

$$T_{break} = -\frac{1}{\min\{h'(x)\}}$$

- If $h'(x) < 0$ at some point then the solution will break and a *shock wave* will form.



(I) $h'(s) \geq 0$ for all s (II) $h'(s_0) < 0$ for some s_0

Indeed, a simple analysis of (**) shows that two characteristics will intersect if and only if the system

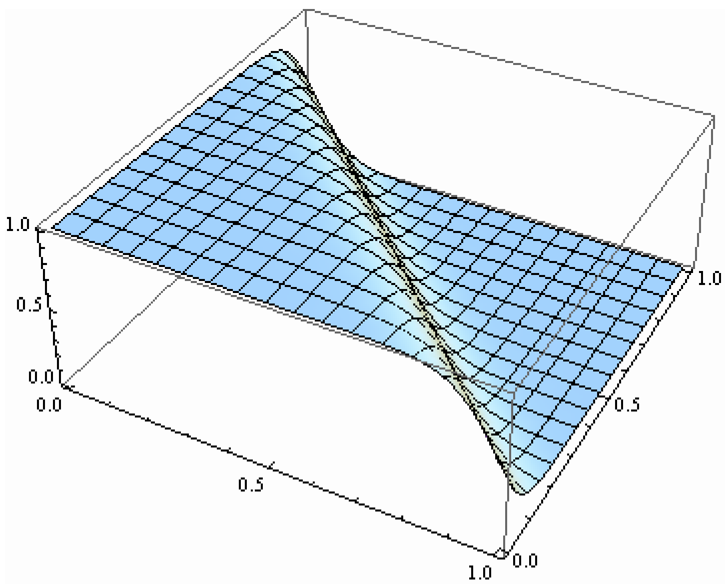
$$x = h(s_1)t + s_1$$

$$x = h(s_2)t + s_2$$

has a *positive* solution *t* (i.e. for positive time), which is equivalent to

$$\tilde{t} = -\frac{s_1 - s_2}{h(s_1) - h(s_2)} > 0$$

If this condition is satisfied for some values s_1 and $s_2$then solution suffers a *gradient catastrophe* type of singularity (otherwise solutions exist globally). An example of the gradient catastrophe**:**

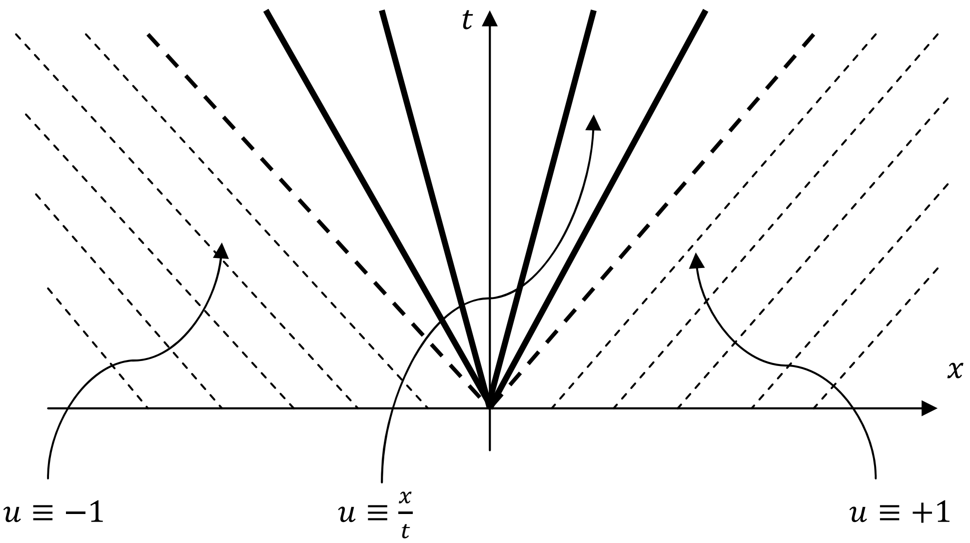


Weak solutions

We consider some instructive examples.

Example 1. Consider the initial data ("running" particles)

$$h(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



Where the central function is the so-called *rarefaction*. Our solution has the form

$$u(x,t) = \begin{cases} -1, & x < 0, \quad 0 < t < -x \\ \dfrac{x}{t}, & |x| \leq t \\ +1, & x > 0, \quad 0 < t < x \end{cases}$$

- This function is continuous

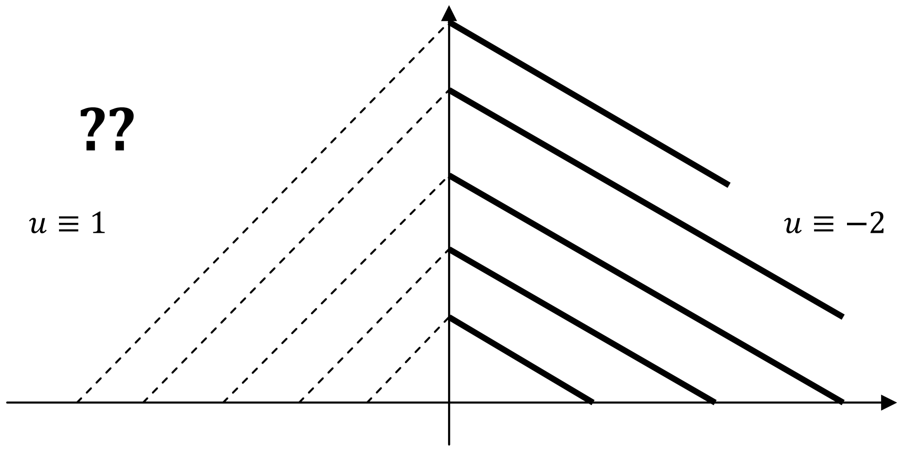
- The found function satisfies the integral equation (prove!):

$$\frac{u(b,t)^2}{2} - \frac{u(a,t)^2}{2} + \frac{d}{dy}\int_a^b u(x,t)\,dx = 0$$

Example 2. Tthe initial data

$$h(x) = \begin{cases} 1 & x < 0 \\ -2 & x \geq 0 \end{cases}$$

We obtain by a similar argument



The characteristics:

$$x = t + x_0 \qquad \text{if } x_0 < 0$$

$$x = -2t + x_0 \qquad \text{if } x_0 \geq 0$$

This is **not** a weak solution: take $a = -1$ and $b = 1$. Then

$$\int_a^b u(x,t)\,dx = 0$$

while

$$\frac{u(b,t)^2}{2} - \frac{u(a,t)^2}{2} = \frac{4-1}{2} = \frac{3}{2} \neq 0$$

Rankine-Hugoniot condition for general conservation laws

The simplest conservation law is the following 1st order PDE:

$$(G(u))'_x + u'_t = 0 \qquad (*)$$

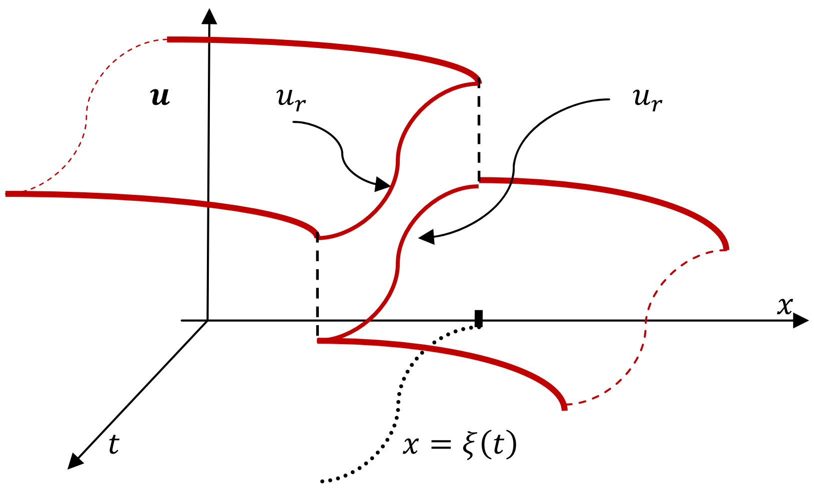
where G is some continuously differentiable function of *u.*

The weak form of (*), also called a conservation law, is the following integrated identity

$$G\big(u(b,t)\big) - G\big(u(a,t)\big) + \frac{d}{dt}\int_a^b u(x,t)\,dx = 0.$$

- A weak solution may contain discontinuities, may not be differentiable, and it require less smoothness than a classical solution.
- If a classical solution to the problem exists, it will also satisfy the definition of a weak solution.

Rankine-Hugoniot condition. We consider the simplest case when discontinuity of *u,* also called a *shock front*, is projected to a smooth curve front $x = \xi(t)$ in the xt-plane. In other words, for a fixed $t > 0$ function u has a jump discontinuity at $x = \xi(t)$:



Then we have the following necessary condition for the shock front $x = \xi(t)$:

Jump condition or the Rankine-Hugoniot condition:

$$\xi'(t) = \frac{G(u_r) - G(u_l)}{u_r - u_l}$$

Here u_l and u_r denote the limiting values of u from the left and right sides of the shock.

■Indeed, taking $\xi(t)$ between a and b we wind by the definition

$$G\big(u(b,t)\big) - G\big(u(a,t)\big) + \frac{d}{dt}\left(\int_a^{\xi(t)} u(x,t)\,dx + \int_{\xi t}^{b} u(x,t)\,dx\right) =$$

$$= G\big(u(b,t)\big) - G\big(u(a,t)\big) + \xi'(t)(u_l(\xi(t),t) - u_l(\xi(t),t)) + \int_a^{\xi(t)} u_t' dx + \int_{\xi(t)}^{b} u_t' dx = 0,$$

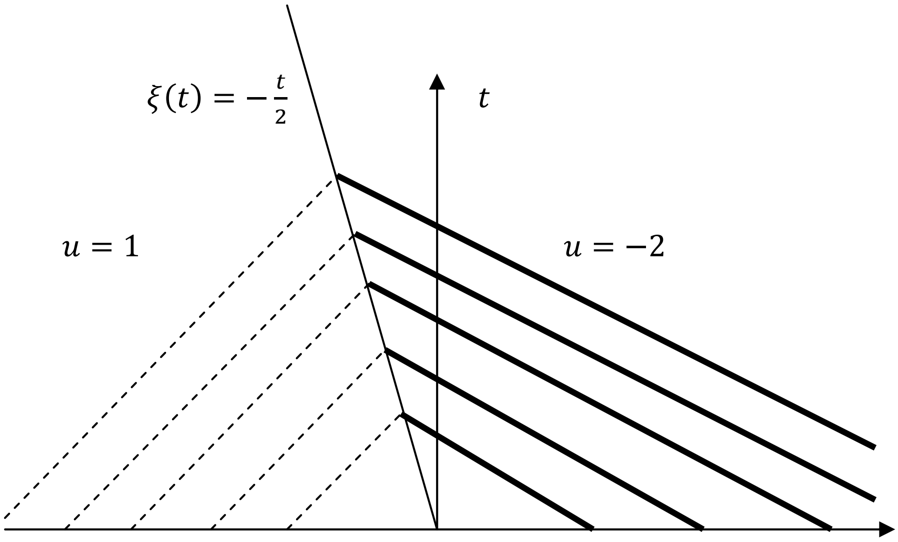
and letting $a \to \xi(t) - 0$ and $b \to \xi(t) + 0$ we get the above relation. ■

In the case of Burger's equation we have $G = \frac{u^2}{2}$, hence the Rankine-Hugoniot condition is

$$\xi'(t) = \frac{u_r + u_l}{2}$$

Return to Example 2: we have $u_r = -2$ and $u_l = 1$, hence

$$\xi'(t) = \frac{u_r + u_l}{2} = -\frac{1}{2} \qquad \Rightarrow \qquad \xi(t) = \xi_0 - \frac{t}{2} = -\frac{t}{2}$$



One can prove that so obtained function will be actually a weak solution to the Burger's equation.