Hamilton Jacobi equation

$$u_t' + H(Du, x) = 0$$

Example 1. Let $H(p, x) = |p|^2 - 1$, then $u(x, t) = t + v(x)$ is a solution to the Hamilton-Jacobi equation if and only if $v(x)$ is a solution to the ikonal equation

$$|Dv| = 1.$$

Example 2. Let $H(p, x) = \frac{|p|^2}{2m} + \frac{kx^2}{2}$. This is the Hamiltonian for a particle on an elastic string (Hooke's Hamiltonian), which is equal to the total *energy*: the sum of kinetic energy $\frac{|p|^2}{2m}$ and potential energy $\frac{kx^2}{2}$. Here p is the momentum of the particle.

In order to find characteristics we notice that our independent variables: $x = (x_1, x_2, \dots, x_n)$ and t.Taking s as a inner parameter along the characteristic curves we obtain famous Hamilton relations for duality between momentum p and coordinate x:

$$\dot{x}_k = H_{p_k}(p, x)$$

$$\dot{p}_k = -H_{x_k}(p, x)$$

and equation for time, $\dot{t} = 1$ and for the unknown function:

$$\dot{u} = \sum_{k=1}^{n} p_k H_{p_k}(p, x) + p_{n+1}$$

- Here $p_{n+1} = u_t'$ and $p = Du = (u_{x_1}', u_{x_2}', \dots, u_{x_n}')$ is the *generalized momentum*
- It follows form the third equation that $t = s$, hence without loss of generality we can consider time t as the inner parameter along the characteristic curves

As in the case of the eikonal and inviscid Burger's equation, in general, there are "few" classical solutions, and our goal is to find a formula for an appropriate weak or generalized solution which exists for all times $t > 0$.

Variational interpretation

Historically, the pair

$$\dot{x}_k = H_{p_k}(p,x), \qquad k = 1 \dots n$$

$$\dot{p}_k = -H_{x_k}(p,x), \qquad k = 1 \dots n$$

of Hamilton-Jacobi equations arises in the classical mechanics as consequence of some calculus of variations.

Dictionary:

$$\boldsymbol{p} = \boldsymbol{Du}$$

$$\boldsymbol{q} = \dot{\boldsymbol{x}}$$

Let us consider an arbitrary Lagrangian, that is a function of q-variables and coordinates:

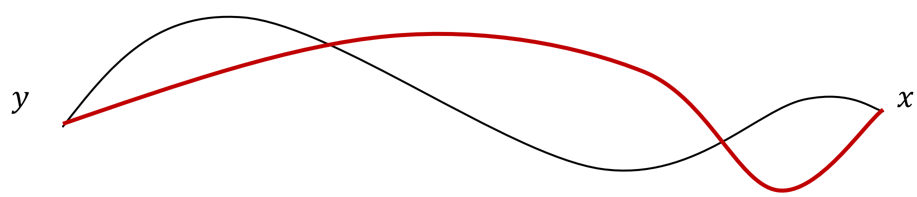
$$L = L(q,x) \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad q = (q_1, q_2, \dots, q_n),$$

is a smooth function and introduce the action (functional)

$$\mathcal{E}(w) = \int_0^t L\big(\dot{w}(s), w(s)\big)\, ds$$

defined for vector-functions $w(s) \colon [0,t] \to \mathbb{R}^n$ in the class

$$A = \{w = w(s) \colon \quad w(0) = y, \;\; w(t) = x\}$$



Euler-Lagrange equations. *The function $X = X(s)$ solves the system of Euler-Lagrange equations:*

$$\frac{d}{ds}\Big(D_q L\big(\dot{X}, X\big)\Big) = D_x L\big(\dot{X}, X\big), \qquad 0 \le s \le t$$

Example 3: Hooke's law. Let us consider the Lagrangian for one-dimensional case: $n = 1$,

$$L(q,x) = \frac{mq^2}{2} - \frac{kx^2}{2}$$

Then the Euler-Lagrange equation is the Hooke law:

$$\frac{d}{ds}\big(m\dot{X}\big) = -kX, \qquad \text{that is} \qquad \ddot{X} = -\frac{k}{m}X$$

Reduction to Hamilton's ODE

We now convert the Euler-Lagrange equations (2nd order ODE) into Hamiltonian's equations (1st order ODE).

In mechanics, the q-derivative of the Lagrangian is called the generalized momentum, hence we define the momentum variables by the system

$$p = D_q L(q, x)$$

We shall assume that $L(q, x)$ satisfies the following property: the above equation allows to express all q as functions of p and x:

$$q = Q(p, x)$$

and the function $Q(p, x)$ is smooth. In other words, we have

$$p = D_q L(Q(p, x), x).$$

Then we can associate with the given Lagrangian $L(q, x)$ the Hamiltonian

$$H(p, x) = \langle p, Q(p, x)\rangle - L(Q(p, x), x).$$

Example 3 (cont). For the Hooke's Lagrangian we have

$$D_q L(q, x) = \left(\frac{mq^2}{2} - \frac{kx^2}{2}\right)_q = mq \equiv p.$$

Hence $q = Q(p, x) = \frac{p}{m}$ and the corresponding Hamiltonian is

$$H(p, x) = \frac{p^2}{m} - \left(\frac{mQ^2(p, x)}{2} - \frac{kx^2}{2}\right) = \frac{p^2}{m} + \frac{kx^2}{2}$$

(cf. Example 2).

Now we rewrite our Euler-Lagrange equations in terms of p and x:

Theorem (Derivation of Hamilton's ODE)**.** *In the introduced notation,*

$$\dot{X}(s) = D_p H(p(s), X(s))$$

$$\dot{p}(s) = -D_x H(p(s), X(s))$$

and $H(p(s), X(s)$ is constant along the integral curves.

Legendre transform and Hopf-Lax formula

$$\begin{array}{ccc} \text{Hamilton-Jacobi PDE} & \longleftrightarrow & \text{Hamilton's ODE} \\ \updownarrow & & \updownarrow \\ \text{Variational interpretation} & \longleftrightarrow & \text{Euler-Lagrange ODE} \end{array}$$

Our conjectures:

- We assume that $H = H(p)$ (that is our situation is homogeneous with respect to space coordinates)

- The Lagrangian under consideration $L(q)$ is also independent of x and will be assumed to be a convex function having superlinear growth at infinity:
$$\lim_{q\to\infty} \frac{L(q)}{|q|} = +\infty$$
- Convexity:
$$L(tq_1 + (1-t)q_2) \leq \; t\, L(q_1) + (1-t)\, L(q_2)$$

- Convexity implies continuity (and even almost everywhere differentiability and a.e. existence the second differential)

The **Legendre transform** of a function $L(q)$ is

$$L^*(p) = \max_{q\in\mathbb{R}^n}\{\, \langle p\,, q\rangle - L(q)\}\,, \qquad p \in \mathbb{R}^n\,.$$

- The maximum is achieved at some point $q^* \in \mathbb{R}^n$ and applying the Fermat theorem we deduce that
$$p = \nabla_q L(q^*)$$
Hence, one can solve the last equation, that is $q^* = Q(p)$

- Moreover, we find
$$L^*(p) = \langle p\,, q^*\rangle - L(q^*) = \langle p\,, Q(p)\rangle - \; L\big(Q(p)\big) = H(p)$$

$$\boxed{H(p) = L^*(p)}$$

Theorem 1. The Legendre transform is an involution:

(i) $H(p)$ is a convex function with superlinear growth, and
(ii) $H^*(q) = L(q)$.

Finally, let us minimize the following action functional:

$$\mathcal{E}_{L,g}(w) = \int_0^t L\big(\dot{w}(s)\big)\, ds + \ g\big(w(0)\big)$$

subject to the conditions: $w(0) = y$ is an arbitrary point in $\mathbb{R}^n$ and $w(t)$ is the given x.

In other words, we define

$$u(x,t) = \inf\left\{\int_0^t L\big(\dot{w}(s)\big)\, ds + \ g\big(w(0)\big), \quad w(0) = y, w(t) = x\right\}$$

The infimum is taken over all C^1-functions w with $w(t) = x$.

Theorem 2. (Hopf-Lax formula) *An explicit form of the above defined function $u(x,t)$ is*

$$u(x,t) = \min_{y \in \mathbb{R}^n}\{t\, L\left(\frac{x-y}{t}\right) + g(y)\}$$

Proof: Denote by U the right hand side in the last formula. Fix $y \in \mathbb{R}^n$ and consider

$$w(s) = y + \frac{s}{t}(x-y)$$

we obtain

$$u(x,t) \le \int_0^t L\big(\dot{w}(s)\big)\, ds + \ g\big(w(0)\big) = t\, L\left(\frac{x-y}{t}\right) + g(y)$$

Hence $u(x,t) \le U$.

In other direction, since L is convex, it satisfies the Jensen's inequality:

$$L\left(\frac{1}{t}\int_0^t \dot{w}(s)\, ds\right) \le \frac{1}{t}\int_0^t L(\dot{w}(s))\, ds.$$

Because $\frac{1}{t}\int_0^t \dot{w}(s)\, ds = \frac{w(t)-w(0)}{t} = \frac{x-y}{t}$, where we denoted $w(0) = y$, we obtain

$$tL\left(\frac{x-y}{t}\right) + g(y) \le \int_0^t L\big(\dot{w}(s)\big) ds + g(y)$$

Hence $U \le \ u(x,t)$. ■

Theorem. Suppose $u(x,t)$ is defined by the Hopf-Lax formula and the initial data is a Lipschitz function:

$$|g(x) - g(y)| \le M\|x-y\|, \qquad \forall x, y \in \mathbb{R}^n$$

Then $u(x,t)$ is Lipschitz continuous in $\mathbb{R}^n \times (0,+\infty)$, and solves the initial value problem

$$u_t' + H(Du,x) = 0, \quad u(x,0) = g(x).$$