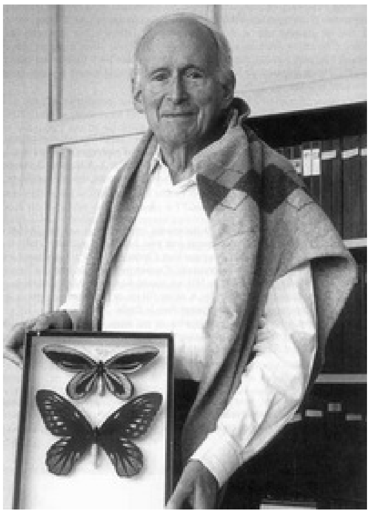
Serguei L'vovich **Sobolev** (06.10.1908 - 03.01.1989).

Generalized functions (later known as distributions), were first introduced by Sobolev in 1935 for weak solutions, and further developed by Laurent Schwartz.

1943-57 participated in the A-bomb project of the USSR. In 1956 Sobolev joined a number of prominent scientists in proposing a large-scale scientific and educational initiative for the Eastern parts of the Soviet Union, which resulted in the creation of the Siberian Division of the Academy of Sciences. He was the founder and first director of the Institute of Mathematics at Akademgorodok near Novosibirsk, which was later to bear his name.

Laurent Schwartz (05.03.1915 - 04.02.2002).



In November 1944 Schwartz discovered distributions: as convolution operators on the space of test functions, not quite their final definition (that was to come to him in February 1945) as continuous linear functionals on space $C_0^\infty(\mathbb{R}^n)$.

One of the "founders" of Bourbaki. In 1950 he was awarded the Fields Medal for his distribution theory. Among other important contributions are homology and cohomology of smooth manifolds and quantum field theory, noncommutative harmonic analysis.

Paul Adrien Maurice Dirac, (08.08.1902 – 20.10.1984)

A British theoretical physicist, made fundamental contributions to the early development of both quantum mechanics and quantum electrodynamics. Nobel Prize in physics for 1933. Among other discoveries, he formulated the Dirac equation, which describes the behavior of fermions and which led to the prediction of the existence of antimatter.

In *Principles of Quantum Mechanics*, published in 1930, the *delta-function* δ is appeared for the first time.

Test functions and distributions

The support of a function f defined in an open subset U of $\mathbb{R}^n$ is the set

$$\operatorname{supp} \varphi = \overline{\{x \in U : f(x) \neq 0\}}.$$

Def. Denote by $\mathcal{D}(U) = C_0^\infty(U)$ the class of all infinitely differentiable functions with compact support in U: $\operatorname{supp} \varphi \ \subset U$. The elements of $\mathcal{D}(U)$ are called also *test functions.* A sequence φ_n converges in $\mathcal{D}(U)$ to a test function $\varphi \in \mathcal{D}(U)$ if there is a compact subset $K \subset U$ such that

$$\operatorname{supp} \varphi_n \subset U, \qquad \forall n = 1,2,3 \ldots$$

and $\|\varphi_n - \varphi\|_{C^m(U)} \to 0$ in for any $m \geq 0$. We write also in this case $\varphi_n \xrightarrow{\mathcal{D}(U)} \varphi$.

Def. A linear functional

$$F : \mathcal{D}(U) \to \mathbb{R}$$

is called a *generalized function*, or a *distribution*, if it is continuous in the sense that $\varphi_n \xrightarrow{\mathcal{D}(U)} \varphi$ implies $F(\varphi_n) \to F(\varphi)$. The (vector) space of all distribution is denoted by $\mathcal{D}'(U)$. Similarly one defines distributions with values in $\mathbb{C}$.

Example 1. Recall that $L^1_{loc}(U)$ denotes the class of locally integrable functions in U, that is functions integrable on any compact subset of U. Then one easily can prove that any locally integrable function $f \in L^1_{loc}(U)$ induces a distribution

$$F_f(\varphi) = \int_U f(x)\varphi(x)dx.$$

In this case, f sometimes is called the symbol of F_f. Usually we identify the distribution F_f and its symbol f.

Any distribution which can be obtained as in Example 1 is called *regular.*

Remark 1. Notice that a regular distribution is uniquely determined by its symbol, that is if $F_f = F_g$ then $f = g$ a.e. in U.

Example 2. The delta-function of Dirac, $\delta_a(\varphi) = \varphi(a)$ $(a \in U)$ is an example of non-regular distribution.

■ The fact that the delta-function is a distribution follows readily from the definition. We show that it is non-regular. Let us argue by contradiction, say, there is a function $f \in L^1_{loc}(\mathbb{R}^n)$ such that

$$\delta_0(\varphi) = \varphi(0) = \int_U f(x)\varphi(x)dx$$

for any $\varphi \in \mathcal{D}(U)$. Consider the function $h(t)$ which is equal to 0 for $t > 1$, given by the formula

$$h(t) = \left(1 - \exp\left(-\frac{1}{t^2}\right)\right) \cdot \exp\left(1 - \frac{1}{(1-t)^2}\right).$$

for $t \in (0,1)$, and equal to 1 for $t = 0$. Then a radial symmetric function $h(|x|)$ is an infinitely differentiable in $\mathbb{R}^n$ and has the compact support

$$\operatorname{supp} h(|x|) = \overline{B_0(1)} = \{x \in \mathbb{R}^n : |x \leq 1|\}.$$

Notice that for any $\varepsilon > 0$ the dilated function $h\left(\frac{|x|}{\varepsilon}\right)$ is a test function:

$$\varphi_\varepsilon(x) = h\left(\frac{|x|}{\varepsilon}\right) \in \mathcal{D}(\mathbb{R}^n).$$

Then we have by the definition $\delta_0(\varphi_\varepsilon) = \varphi_\varepsilon(0) = 1$ and $|\varphi_\varepsilon(x)| \leq 1$. Hence,

$$1 = \left|\int_{B_0(\varepsilon)} f(x)\varphi_\varepsilon(x)dx\right| \leq \int_{B_0(\varepsilon)} |f(x)|\, dx,$$

because $\operatorname{supp}\varphi_\varepsilon = \overline{B_0(\varepsilon)}$. The function f is integrable on $B_0(\varepsilon)$ and letting $\varepsilon \to 0$ we conclude that the latter integral converges to zero. The contradiction follows. ■

Example 3. It can be shown (see lecture notes) that the functional

$$G(\varphi) = \lim_{\varepsilon \to 0}\left(\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\, dx + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x}\, dx\right)$$

is a distribution.

Product of a distribution $F \in \mathcal{D}'(U)$ and a smooth function $h \in C^\infty(U)$ is a new distribution defined by

$$(hF)(\varphi) = F(h\varphi).$$

Then for regular distributions we have $hF_g = F_{gh}$ (for $h \in C^\infty(U),\ g \in L^1_{loc}(U)$) which motivates the definition.

Derivative of a distribution

Let us consider a regular distribution F_f in U with a smooth symbol, say, $f \in C^m(U), m \geq 1$. Then for any test function $\varphi \in \mathcal{D}(U)$ we have (by integrating by parts)

$$F_f(D^\alpha\varphi) = \int_U f\, D^\alpha\varphi\, dx = (-1)^{|\alpha|} \int_U \varphi\, D^\alpha f dx = (-1)^{|\alpha|} F_{D^\alpha f}(\varphi)$$

Taking into account our identification between a symbol and the associate regular distribution, the following definition is a natural generalization of derivative.

Def. For any $F \in \mathcal{D}'(U)$ we define the (weak) derivative $D^\alpha F$ (α is some multi-index) by the following rule:

$$(D^\alpha F)(\varphi) = (-1)^{|\alpha|}\, F(D^\alpha\varphi).$$

It easy to see that this new functional is actually a distribution: $D^\alpha F \in \mathcal{D}'(U)$.

Hence any generalized function becomes infinitely differentiable in the above sense. In particular, regarding any function in $L^1_{loc}(U)$ as a symbol of the associate regular distribution, we see that the weak derivative of such a function is defined at least on the level of distribution. Sometimes the weak derivative is also a regular functional. This motivates the following definition.

Def. If both the distribution $F \in \mathcal{D}'(U)$ and its derivative $D^\alpha F$ are regular then one says on the existence of the *weak derivative* (D^α) of the associate symbol. That is, if $F = F_g$ for some $g \in L^1_{loc}(U)$ and $D^\alpha F = F_h$, for some $h \in L^1_{loc}(U)$ then h is called the weak D^α-derivative of g. In other words, a function $g \in L^1_{loc}(U)$ has a weak D^α-derivative if there is $h \in L^1_{loc}(U)$ such that

$$\int_U g\, D^\alpha\varphi dx \;=\; (-1)^{|\alpha|} \int_U \varphi h\, dx\,, \qquad \forall\varphi \in \mathcal{D}(U).$$

One denotes the weak derivative also by $D^\alpha g$. As a corollary of Remark 1, the weak derivative of a locally integrable function, if well-defined, is uniquely determined.

Fundamental solution

Now we can also define action of a linear **partial differential operator** (PDO) on distributions. Consider an operator of order m

$$L\varphi = \sum_{|\alpha|\leq m} a_\alpha\, D^\alpha\varphi$$

with smooth coefficients a_α. Then it is natural to define action of L on distributions by

$$LF = \sum_{\alpha} a_\alpha D^\alpha F\,, \qquad F \in \mathcal{D}'(U).$$

Hence, by the definition

$$(LF)(\varphi) = \sum_{\alpha} a_\alpha\, D^\alpha F\,(\varphi) = \sum_{\alpha} D^\alpha F\,(a_\alpha \varphi) = \sum_{\alpha} (-1)^{|\alpha|}\, F(D^\alpha\,(a_\alpha \varphi))$$
$$= F\left(\sum_{\alpha} (-1)^{|\alpha|} D^\alpha (a_\alpha\, \varphi)\right)$$

The operator

$$L'\varphi = \sum_{\alpha} (-1)^{|\alpha|} D^\alpha (a_\alpha\, \varphi)$$

sometimes is called the *adjoint* operator to L. This, we obtain

$$LF = FL'.$$

Definition. A distribution $E = E(a)$ is called the fundamental solution of L if E is a solution (in the distributional sense) of the inhomogeneous equation

$$LE = \delta_a.$$

Examples of weak derivatives

Example 4. The weak derivative of the delta-function in the one-dimensional case is

$$\frac{\partial}{\partial x}\delta_a(\varphi) \equiv \delta_a'(\varphi) = -\delta_a\left(\frac{\partial \varphi}{\partial x}\right) = -\,\frac{\partial \varphi}{\partial x}(0).$$

Example 5. Let h be the Heaviside function: $h(x) = 1$ for $x \geq 0$ and $h(x) = 0$ for negative $x < 0$. Take $\varphi \in \mathcal{D}(\mathbb{R})$ and find $M > 0$ such that $\operatorname{supp} \varphi \subset [-M, M]$. Then

$$h'(\varphi) = -\int_{-\infty}^{+\infty} \varphi' h\, dx = -\int_0^M \varphi'\, dx = \varphi(0) = \delta_0(\varphi).$$

Hence, the weak derivative of the Heaviside function is the delta-function centered at zero. Another interpretation of this property is that (due to definition above) *the shifted Heaviside function* $h(x-a)$ *is the fundamental solution of the first order operator* $L = \frac{d}{dx}$.

Example 6 (Two-dimensional Heaviside function). Consider now the following step-function

$$h(x_1, x_2) = \begin{cases} 1 & x_1, x_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $= \frac{\partial^2}{\partial x_1 \partial x_2}$. For any $\varphi \in \mathcal{D}(\mathbb{R}^2)$ we find $M > 0$ such that $\operatorname{supp} \varphi \subset [-M, M] \times [-M, M]$. We have

$$Dh\,(\varphi) = \int_{\mathbb{R}^2} h\, D\varphi dx = \int_0^M dx_1 \int_0^M \frac{\partial^2 \varphi}{\partial x_1 \partial x_2}\, dx_2 = -\int_0^M \varphi'_{x_1}(x_1, 0) dx_1 = \varphi(0,0).$$

Hence h is the fundamental solution of the operator $L = \frac{\partial^2}{\partial x_1 \partial x_2}$. Notice that the latter operator is the normal form of the wave operator. Hence, the fundamental solution of the wave operator in two dimensions is a regular distribution (locally integrable function) $h(x_1 - a_1, x_2 - a_2)$. It turns out that it is not true for higher dimensions.

Example 4. The Laplace operator in $\mathbb{R}^n$ is self-adjoint: $\Delta = \Delta'$. One can show (see lecture notes) that the function

$$\Psi(x) = \begin{cases} \frac{1}{2\pi} \ln|x| & n = 2 \\ -\frac{1}{(n-2)\omega_n |x|^{n-2}} & n \geq 3 \end{cases}$$

is the fundamental solution of the n-dimensional Laplacian in the weak sense.

Remark 2. An important property of the above function $\Psi(x)$ is that it is actually is a local integrable function in $\mathbb{R}^n$. It follows from a general fact that $|x|^p$ is locally integrable in $\mathbb{R}^n$ for $p + n > 0$.