5. Sobolev spaces: basic definitions

5.1 Auxiliary facts

We shall use the following well-known facts about completion of a normed space.

Let X, Y be two vector spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. A linear map $T: X \to Y$ is called an *isometry* if $\|Tx\|_Y = \|x\|_X$ for any $x \in X$. As a corollary, an isometry is always an injection and one usually identify X and $T(X) \subset Y$.

Theorem (completion of a normed space).

Every normed space $(X, \|\cdot\|_X)$ embeds isometrically as a dense subspace of a Banach space $(Y, \|\cdot\|_Y)$. That is there is a Banach space $(Y, \|\cdot\|_Y)$ and a linear isometry $i: X \to Y$ such that $i(X)$ is dense in Y.

Usually one denotes the completion as a closure: $Y = \bar{X}$

Remark. The canonical completion can be defined even for general metric spaces. It has the structure of the factor space $X_{Cauchy}/\sim$, where X_{Cauchy} is the (vector) space of all Cauchy sequences in X w. r. t. the norm $\|\cdot\|_X$. The equivalence relation $\sim$ is defined by: $(x_n) \sim (x_n')$ iff $\lim_{n\to\infty}\|x_n - x_n'\|_X = 0$. The canonical embedding is then the map $i(x) = (x, x, x, \dots) \in X_{Cauchy}$ which maps a point $x \in X$ to the constant sequence.

Examples.

a) The real numbers are completion of rational numbers $\overline{\mathbb{Q}} = \mathbb{R}$ with respect to the absolute value norm $\|x\| = |x|$
b) The completion of rational numbers with respect to the norm $\|x\| = |x|_p$ is $\overline{\mathbb{Q}} = \mathbb{Q}_p$ ($|x|_p$ is the so called *p-adic norm* defined for a given prime $p \geq 2, p \in \mathbb{N}$ and for a non-zero rational number $x = \frac{p^k a}{b}$ as $|x|_p = p^{-k}$, where a, b are coprime to p).
c) The Lebesgue p-space

$$L^p(U) = \{f(x) \text{ measurable and } \int_U |f(x)|^p\, dx < \infty\}, \qquad 1 \leq p < \infty$$

can be equivalently characterized as the completion of space $C_0^\infty(U)$ of test functions with respect to the integral norm

$$\|\varphi\|_p \equiv \|\varphi\|_{L^p(U)} = \left(\int_U |\varphi(x)|^p dx\right)^{1/p}$$

Notice that for $p = 1$, a linear functional $I(\varphi) = \int_U \varphi(x)dx$ is a bounded functional:

$$I(\varphi) \leq \|\varphi\|_1,$$

hence it has a unique extension to a bounded functional on $L^1(U)$. This extension is well-known as the Lebesgue integral.

In examples (a) and (c) (for $p = 2$) the norms are generated by corresponding inner (scalar) products.

Equivalent norms. Two norms $\|x\|_1$ and $\|x\|_2$ on X are equivalent if they which satisfy the bilateral inequality $M_1\|x\|_1 \leq \|x\|_2 \leq M_2\|x\|_1$, with $M_k > 0$. Then topologically $(X, \|x\|_1)$ and $(X, \|x\|_2)$ (as well as their completions) are homeomorphic (with a homeomorphism given by the identity operator).

In what follows, we identify two (normed) spaces with equivalent norms.

Another important fact is the following

Theorem. *If $(X, \langle\cdot,\cdot\rangle)$ is a space with the scalar product $\langle\cdot,\cdot\rangle$ then the completion $\bar{X}$ is a scalar product space; the scalar product restriction to X coincides with $\langle\cdot,\cdot\rangle$. Moreover, $\bar{X}$ is a Hilbert space with respect to the mentioned extension of $\langle\cdot,\cdot\rangle$.*

5.2 Sobolev spaces: basic definitions

The number of continuous derivatives measures how regular the function is. We have the following chain of classes of regularity:

$$L^1_{loc}(U) \supset C^0(U) \supset \mathrm{Lip}(U) \supset \boldsymbol{C^1(U)} \supset \boldsymbol{C^2(U)} \supset \cdots \supset \boldsymbol{C^\infty(U)} \supset \boldsymbol{C^\omega(U)} \supset \mathbf{Poly}$$

In order to work with weak solutions one has to relax the notion of derivative and understand it in an "integral" sense. For any domain U in $\mathbb{R}^n$ we define the scalar product

$$\langle u, v\rangle_1 = \langle u, v\rangle_{H^{1,2}(U)} = \int_U (\nabla u \cdot \nabla v + uv)\, dx$$

and the associated norm

$$\|u\|_{1,2} = \sqrt{\langle u, u\rangle_1} = \left(\int_U (|\nabla u|^2 + u^2)\, dx\right)^{1/2} = \left(\|u\|_2^2 + \sum_{k=1}^{n} \left\|u'_{x_k}\right\|_2^2\right)^{1/2}$$

where u, v are functions of class $C^1(U)$ (or class $C^1_0(U)$).

Remark 1. Adding the L^2-norm of the gradient $\|\nabla u\|_2$ has effect of a "smoothness" on L^2. In other words, one can canonically define the derivatives for functions in the completed space. We shall see this later.

Remark 2. The introduced $\|u\|_{1,2}$ satisfies all the properties of a norm on any subspace of differentiable functions. In particular, if we consider all differentiable functions with continuous derivative we obtain a normed vector space $(C^1(U), \|\cdot\|_{1,2})$. Unfortunately, this (normed) space is not complete as shows the next example.

Example 1. Consider the sequence

$$f_k(x) = \begin{cases} 1 - \frac{1}{2n} - \frac{nx^2}{2}, & |x| < \frac{1}{n} \\ 1 - |x|, & |x| \geq \frac{1}{n} \end{cases}$$

is a Cauchy sequence (in $\|\cdot\|_{1,2}$) in the interval $U = [0,1]$, while it has no limit in $C^1([-1,1])$.

Indeed, one can show (check!) that for $1 \leq m \leq n$

$$\|f_n - f_m\|_{1,2}^2 = \frac{(20n^2m^2 + 3n^2 + 6nm - m^2)(m-n)^2}{60n^3m^4}$$

Hence (f_n) is a Cauchy sequence, while the pointwise limit (which is a strong limit in the $L^2([-1,1])$-norm) is function $f = |x|$. One can see that $f \notin C^1([-1,1])$. ■

Definition. Define the Sobolev space $H_0^{1,2}(U)$ as the completion of $C_0^1(U)$ with respect to the norm $\|\cdot\|_{1,2}$. Similarly, $H^{1,2}(U)$ denotes the completion of $C^1(U)$ in the same norm.

Remark 3. $H_0^{1,2}(U)$ and $H^{1,2}(U)$ are Hilbert spaces with the corresponding scalar products induced by $\langle u, v\rangle_1$. Intuitively, one can think of functions in $H_0^{1,2}(U)$ as functions with zero boundary values on ∂U.

Lemma 1. *$H_0^{1,2}(U)$ is a subspace of $L^2(U)$.*

■Indeed, let $(f_n) \in C_0^\infty(U)$ be a Cauchy sequence in $\|\cdot\|_{1,2}$. Then

$$\|f_n - f_m\|_2 \leq \|f_n - f_m\|_{1,2}$$

which implies that (f_n) converges in $L^2(U)$ to some function $f \in L^2(U)$. In particular[1],

$$\overline{C_0^\infty(U)} \subset L^2(U). \qquad ■$$

Thus we have

$$H_0^{1,2}(U) \hookrightarrow L^2(U)$$

and the operator of inclusion is continuous in the usual sense.

Similarly, the sequence of **partial derivatives** $(D_{x_k} f_n)$ form a Cauchy sequence in $L^2(U)$ for any $1 \leq k \leq n$. It follows that for any such k, the derivative $D_{x_k} f_n$ converges in $L^2(U)$ to some function $g_k \in L^2(U)$. It is natural to think of the g_k as the **generalized derivative** of f with respect to x_k in the described integral sense.

[1] The completion with respect to $\|\cdot\|_{1,2}$.

Lemma 2. *In the above notation, the weak derivative $D_{x_k}f$ (in the sense of distributions) is a regular distribution which belongs to class $L^2(U)$. Moreover, $D_{x_k}f = g_k$.*

■ Consider some $k \in \{1,2,\dots,n\}$ and any test function $\varphi \in C_0^\infty(U)$. Since $D_{x_k}f_n \to g_k$ in $L^2(U)$,

$$\left|\int_U \varphi\, D_{x_k}f_n\, dx - \int_U \varphi g_k\, dx\right| \le \left(\int_U \varphi^2\, dx\right)^{1/2} \left(\int_U (D_{x_k}f_n - g_k)^2\, dx\right)^{1/2} \to 0.$$

It follows that

$$\exists \lim_{n\to\infty} \int_U \varphi\, D_{x_k}f_n\, dx = \int_U \varphi g_k\, dx$$

On the other hand, f_n is differentiable, hence

$$\int_U \varphi\, D_{x_k}f_n\, dx = -\int_U f_n\, D_{x_k}\varphi\, dx$$

Since $f_n \to f$ in $L^2(U)$, the latter identity yields

$$\lim_{n\to\infty} \int_U \varphi\, D_{x_k}f_n\, dx = -\lim_{n\to\infty} \int_U f_n\, D_{x_k}\varphi\, dx = -\int_U f\, D_{x_k}\varphi\, dx$$

This proves that

$$\int_U \varphi g_k\, dx = -\int_U f\, D_{x_k}\varphi\, dx,$$

that is g_k is the weak D_{x_k}-derivative of f. ■

The last lemma suggests that a more general Sobolev space may be defined. Namely, we have the following definition.

Definition. We denote by $W^{1,2}(U)$ the subspace of $L^2(U)$ consisting of functions whose weak derivatives are functions in $L^2(U)$. In other words, $W^{1,2}(U)$ consists of functions $f \in L^2(U)$ such that for any $k = 1,\dots,n$ there is $g_k \in L^2(U)$ and

$$\int_U \varphi v_k dx = -\int_U u\varphi'_{x_k} dx$$

for any test function $\varphi \in C_0^\infty(U)$.

Corollary 1. *We have the following embeddings:*

$$H_0^{1,2}(U) \hookrightarrow H^{1,2}(U) \hookrightarrow W^{1,2}(U) \hookrightarrow L^2(U).$$

Moreover, each inclusion is a bounded operator with respect to the corresponding norms.

Remark 4. In fact, it can be shown that $H^{1,2}(U) = W^{1,2}(U)$.

We already know that all Lebesgue spaces $L^p(U)$ are complete. It is natural to extend the above definitions on these classes.

Definition. In general we define

$$\|u\|_{1,p} = \left(\int_U (|\nabla u|^p + u^p)\, dx\right)^{1/p}, \qquad 1 \le p < \infty.$$

By the Minkowski inequality, the latter is a norm on $C_0^1(U)$ and $C^1(U)$, that is

$$\|u + v\|_{1,p} \le \|u\|_{1,p} + \|v\|_{1,p}.$$

Hence we can similarly define $H_0^{1,p}(U)$ and $H^{1,p}(U)$ as the completions with respect to the $\|\cdot\|_{1,p}$ of $C_0^1(U)$ and $C^1(U)$ respectively.

In general W^p-Sobolev spaces are defined then as:

$$W^{1,p}(U) = \{u \in L^p(U)\colon \text{ the weak derivatives } D_{x_k}u \text{ belong to class } L^p(U)\}$$

Finally, one defines the general Sobolev spaces analogously for **higher derivatives**. Namely, for a given integer $m \ge 1$ and a real $p \ge 1$ we define $W^{m,p}(U)$ as space of all $L^p(U)$-functions whose weak derivatives of order $|\alpha| \le m$ are also in $L^p(U)$:

$$W^{m,p}(U) = \{u \in L^p(U)\colon \ D^\alpha u \in L^p(U), \qquad |\alpha| \le m\}$$

The norm in $W^{m,p}(U)$ is then defined by

$$\|u\|_{m,p} = \left(\int_U \left(\sum_{|\alpha|\le m} |D^\alpha u|^p\right) dx\right)^{1/p} = \left(\sum_{|\alpha|\le m} \|D^\alpha u\|_{L^p}^p\right)^{1/p}$$

It is easy to show that this norm is equivalent to

$$\|u\|'_{m,p} = \sum_{|\alpha|\le m} \|D^\alpha u\|_{L^p}$$

Lemma. *$W^{m,p}(U)$ is a Banach space.*

Proof follows is evident.

We comment briefly some **special cases**:

- $W^{1,1}(\Delta) = AC(\Delta)$, where Δ is an interval in $\mathbb{R}$ and $AC(\Delta)$ is class of absolutely continuous functions in Δ

- $W^{1,\infty}(\Delta) = \text{Lip}\ (\Delta)$ is class of Lipschitz functions in the interval Δ

- $W^{m,\infty}(U)$ are normed algebras (product of two elements is again an element of the space)

- $W^{m,2}(U)$ are Hilbert subspaces of $L^2(U)$ for any integer $m \geq 1$ and sometimes they are denoted also by $H^m(U)$

5.3 Examples

Example 1. Modifying the argument given in Example 1 one can show that function $f(x) = 1 - |x|$ on $[-1,1]$ and $f(x) = 0$ otherwise is in $H_0^{1,2}([-1,1])$.

Example 2. Consider $f(x_1, x_2) = a(x_1) + b(x_2)$ such that a and b are arbitrary measurable functions (for example they may be nowhere differentiable). Then the weak derivative $\partial^2_{x_1x_2} f$ exists and equal to zero. Indeed, we choose M large enough such that the support of a test function φ is contained in $[-M, M] \times [-M, M]$. Then

$$\int_{-M}^{M} dx_1 \int_{-M}^{M} f\partial^2_{x_1x_2}\varphi\, dx_2 = \int_{-M}^{M} a(x_1)\, dx_1 \int_{-M}^{M} \partial^2_{x_1x_2}\varphi\, dx_2 + \int_{-M}^{M} b(x_2)\, dx_2 \int_{-M}^{M} \partial^2_{x_1x_2}\varphi\, dx_1$$

where the inner integrals in the right hand side are equal to zero. Hence $\partial^2_{x_1x_2} f = 0$ in the distributional sense.

The following fact is useful in our further applications.

Lemma. *For any* $R > 0$

$$|x|^k \in W^{m,p}(B_0(R)) \quad \textit{if and only if} \quad k < \frac{n}{p} - m$$

and the weak derivatives coincide with the usual derivatives of $|x|^k$ *in* $B_0(R)\backslash\{0\}$.

■Denote $B_{0(R)} = B$ and notice that

$$|x|^{-k} \in L^p(B) \quad \text{if and only if} \quad pk < n. \qquad (*)$$

Consider any test function $\varphi \in C_0^\infty(B)$ and fix some $k < \frac{n}{p}$. Then $n - k > n - pk > 0$ (since $p \geq 1$) and

$$\int_B \varphi'_{x_i} |x|^{-k}\, dx = \lim_{\varepsilon \to 0} \int_{B \setminus B_\varepsilon} \varphi'_{x_i} |x|^{-k}\, dx$$

because $\varphi'_{x_i} |x|^{-k}$ is integrable for any $i = 1, \dots, n$. Integrating by parts yields

$$\int_{B \setminus B_\varepsilon} |x|^{-k} \varphi'_{x_i}\, dx = \int_{B \setminus B_\varepsilon} \left[\left(\frac{\varphi}{|x|^k} \right)'_{x_i} + \frac{k x_i \varphi}{|x|^{k+2}} \right] dx \; = -\int_{\partial B_\varepsilon} \varphi \frac{\langle e_i, \nu \rangle}{|x|^k} + \int_{B \setminus B_\varepsilon} \frac{k x_i \varphi}{|x|^{k+2}}$$

We have also

$$\left| \int_{\partial B_\varepsilon} \varphi \frac{\langle e_i, \nu \rangle}{|x|^k} \right| \leq \omega_n \max_{\partial B_\varepsilon} |\varphi| \;\cdot\; \varepsilon^{n-k-1}$$

(here ω_n stands for the $(n-1)$-dimensional measure of the unit sphere in $\mathbb{R}^n$). It follows that the latter integral converges to zero as $\varepsilon \to 0$ for $n - k > 1$ and may diverge for $n - k \leq 1$ if $\varphi(0) = 1$. Thus, we have for $n > k + 1$:

$$\int_B \varphi'_{x_i} |x|^{-k}\, dx = \int_B \frac{k x_i \varphi}{|x|^{k+2}}$$

It follows that in the made assumptions, $D_{x_i}(|x|^{-k}) = \frac{k x_i}{|x|^{k+1}}$ in the weak sense. This yields that $|D(|x|^{-k})| = \frac{k}{|x|^{k+1}}$. This function belongs to $L^p(B)$ for $p(k+1) < n$.

Summarizing, we conclude that $|x|^k \in W^{1,p}(B_0(R))$ if $k < \frac{n}{p} - 1$. The general inclusion $|x|^k \in W^{m,p}(B_0(R))$ follows by induction. ■