6. Sobolev spaces: elementary elliptic equations

6.1 Weak solutions of the Poisson equation

Now we are ready to demonstrate the usefulness of Sobolev spaces in some simple situations. We start with weak solutions of the Poisson equation.

Let U is a bounded open subset of $\mathbb{R}^n$ and let us consider the Dirichlet problem for the Poisson equation:

$$\Delta u(x) = f(x), \qquad x \in U$$

$$u(x) = 0, \qquad x \in \partial U.$$

Since we are interested in the zero boundary condition, it is **reasonable** to consider $C_0^\infty(U)$ as the class of test functions.

Let u be the classical solution of the above Dirichlet problem. Then multiplying the Poisson equation by an arbitrary test function $v \in C_0^\infty(U)$ and integrating the obtained equality by parts yields

$$-\int_U fv\,dx = \int_U \nabla u \cdot \nabla v\,dx \tag{1}$$

The following remarks are appropriate to mention here:

(a) The latter identity may also be interpreted for any function u of a larger class $H_0^{1,2}(U)$.

(b) A bilinear form

$$(u, v) := \int_U \nabla u \cdot \nabla v\,dx$$

in the left hand side looks like a **scalar product**.

(c) The left hand side of (1) can be interpreted as a linear functional (actually, a distribution)

$$F(v) := -\int_U fv\,dx \tag{2}$$

and the scalar product (u, v) resembles then the Riesz representation theorem

(d) If $f \in L^2(U)$, the linear functional $F(v)$ defined above is **bounded** in $L^2(U)$-norm for subspace $C_0^\infty(U)$ and moreover, we have

$$|F(v)| \le \left(\int_U f^2\,dx\right)^{1/2} \left(\int_U v^2\,dx\right)^{1/2} = \|f\|_{L^2}\|v\|_{L^2} \le \|f\|_{L^2}\|v\|_{H_0^{1,2}}$$

that is $F(v)$ extends (by Hahn-Banach theorem) to a bounded functional in $H_0^{1,2}(U)$.

We return to the scalar product (u,v) defined above. In fact, this bilinear form is well-defined in the whole $H_0^{1,2}(U)$, symmetric and positive there (notice that $|u|_1 = 0$ yields $u \equiv const$, which implies, by zero boundary condition, $u \equiv 0$). Denote by

$$|u|_1 = \sqrt{(u,u)}$$

the associated norm.

Hence we arrive at a **weak formulation** of (1): given a function $f \in L^2(U)$, any classical solution u to the Poisson equation with zero boundary condition is a solution of the following problem

$$(u,v) = F(v), \qquad \forall\, v \in C_0^\infty(U), \tag{3}$$

where F is defined by (2). The following is then natural.

Definition. A function $u \in H_0^{1,2}(U)$ which solves (3) is called a *weak solution* to the Poisson equation with zero boundary conditions.

In order to treat the existence of a weak solution we show that the new scalar product generates the same metric (and topological) structure in $H_0^{1,2}(U)$. We prove first an auxiliary result.

6.2 Poincare's inequality

Theorem (Poincare's inequality)[1]. *If U is a bounded open subset of $\mathbb{R}^n$ then there exists a constant $C = C(U) > 0$ such that*

$$\int_U |\nabla v|^2(x)\, dx \geq C \int_U v^2(x)\, dx$$

for all $v \in C_0^1(U)$. Moreover, the inequality holds for any function in $H_0^{1,2}(U)$

■Denote by the same letter v the function obtained by extending v by zero outside of U. Let M be large enough such that the support of v is contained in the cube $Q = \{x : |x_k| \leq M\}$. Notice that $\frac{\partial}{\partial x_k} x_k = 1$, hence integrating by parts we find

$$\int_U v^2(x)\, dx = \int_Q v^2 \frac{\partial x_k}{\partial x_k}\, dx = -2 \int_Q x_k v v'_{x_k} dx \leq 2M \int_Q |v v'_{x_k}| dx$$

Applying Cauchy inequality $\left(\int_Q |v v'_{x_k}| dx\right)^2 \leq \int_Q v^2 dx \int_Q {v'_{x_k}}^2 dx$ we obtain

[1] Sometimes is called also the Fridrichs-Poincare inequality

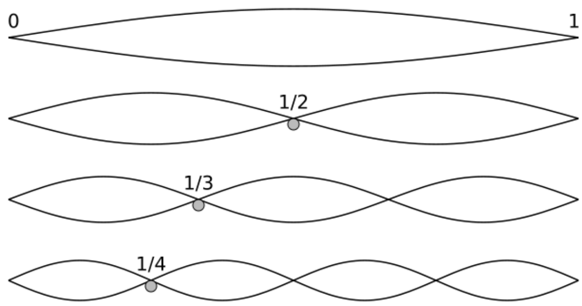
$$\frac{1}{4M^2}\int_U v^2 dx = \frac{1}{4M^2}\int_Q v^2 dx \le \int_Q {v'_{x_k}}^2 dx \le \int_U |\nabla v|^2\,dx$$

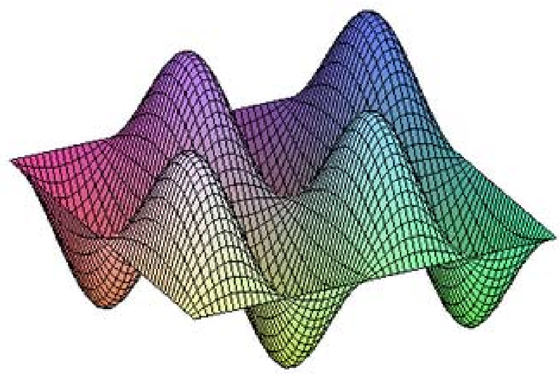
and the desired property follows.■

Remark 1. The question on the optimal constant in the Poincare inequality has many relations to other problems in mathematics and mathematical physics. For instance, if one think of U as a membrane then the best constant $\sqrt{C}$ in the Poincare inequality is exactly the *fundamental frequency* λ_1 (sometimes is called also the fundamental tone) of U.

If $U = [0, L]$ is the one-dimensional interval then $\lambda_1 = \pi/L$ and the corresponding Poincare inequality is called also Wirtinger's inequality.

Besides the fundamental tone, one distinguish also a series of higher "tones" (see the first picture below). The second picture shows the oscillating two-dimensional membrane corresponding to the 7th tone $v = \sin 2x_1 \sin 4x_2$ in the rectangle $U = [0,1] \times [0,1]$.





Remark 2. The above Poincare inequality is a partial case of a more general relation (the so-called Poincare-Friedrichs inequality)

$$\int_U |\nabla v|^p(x)\,dx \ge C_p \int_U v^p(x)\,dx\,, \qquad \forall v \in C_0^1(U)$$

which can be proved by a similar argument.

Corollary 1. *If U is a bounded open subset of $\mathbb{R}^n$ then there exists a constant $C = C(U) > 0$ such that for any $v \in H_0^{1,2}(U)$*

$$\|v\|_{1,2} \le C\|\nabla v\|_2$$

■ Indeed, for any $v \in H_0^{1,2}(U)$ Poincare's inequality yields

$$\|v\|_{1,2}^2 = \int_U (|\nabla v|^2 + |v|^2)dx \le \big(1 + C(U)\big)\int_U |\nabla v|^2\,dx = |v|_1^2 \quad ■$$

Corollary 2 (Existence and uniqueness of the weak solution). *For any* $f \in L^2(U)$ *there is a unique function* $u \in H_0^{1,2}(U)$ *which solves (3).*

■ By Corollary 1 we have $\|v\|_{1,2} \le C_1 \|\nabla v\|_2$ and, in the other direction,

$$|v|_1^2 = \int_U |\nabla v|^2 \, dx \le \int_U (|\nabla v|^2 + |v|^2) dx = \|v\|_{1,2}^2.$$

It follows that $|u|_1$ and $\|u\|_{1,2}$ are ***equivalent norms***.

On the other hand, F defined by (2) is a bounded linear functional in $H_0^{1,2}(U)$. Hence, applying the Riesz representation theorem to the new scalar product (u, v) we immediately conclude that there is a unique function $u \in H_0^{1,2}(U)$ such that

$$F(v) = (u, v), \qquad \forall v \in H_0^{1,2}(U)$$

That is u is a weak solution to the Dirichlet problem formulated above. ■

6.3 Existence of weak solutions for general elliptic equations

The method described above can be applied also to the Dirichlet problem for more general elliptic equations. Consider a differential operator in divergence form

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j}\right) + c(x)u(x).$$

where $a_{ij}(x) = a_{ji}(x)$, $a_{ji}(x) \in C^\infty(\overline{U})$, $c(x) \in C^\infty(\overline{U})$ and $a_{ij}(x)$ satisfying the (uniform) *ellipticity condition*: there is a constant $\alpha > 0$ such that for any $\xi \in \mathbb{R}^n$

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \ge \alpha \|\xi\|^2.$$

Repeating the argument in the previous section, we see that the following definition of a weak solution is natural.

Definition. A function $u \in H_0^{1,2}(U)$ which solves (3) is called a *weak solution* to $Lu = 0$ with zero boundary conditions if

$$-\int_U fv \, dx = \int_U \left(\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} - c(x)uv\right) dx\,, \qquad \forall v \in C_0^\infty(U). \tag{4}$$

(minus before $c(x)$ is due to the odd number of integrating by parts).

It is then convenient to restate the condition for a weak solution as follows:

$$F(v) = B(u, v), \qquad v \in C_0^\infty(U), \tag{5}$$

where $F(\cdot)$ s defined as before by formula (2) and the new bilinear form is given by

$$B(u,v) = \int_U \left(\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} - c(x)uv\right) dx.$$

The existence of a weak solution will follow immediately if we show that $B(u,v)$ is a scalar product whose associated norm is equivalent to the standard norm in $H_0^{1,2}(U)$. So we need to investigate under which conditions $B(u,v)$ meets these criteria.

First we notice that $B(u,v)$ is symmetric by our assumption on the coefficients $a_{ij}(x)$. Next, by using of the ellipticity condition we obtain the following lower estimate:

$$B(u,u) \geq \int_U (\alpha\|\nabla u\|^2 - \gamma u^2)dx \geq \alpha\|\nabla u\|_2^2 - \gamma\|u\|_2^2,$$

where $\gamma = \max\{c(x): x \in U\}$. Let us assume that

$$\gamma = \max\{c(x): x \in U\} < \alpha C$$

where C is the constant in the Poincare inequality. If $\gamma \leq 0$ then $B(u,u) \geq \alpha\|\nabla u\|_2^2$, and if $\gamma > 0$ then

$$B(u,u) \geq \alpha\|\nabla u\|_2^2 - \gamma\|u\|_2^2 > \left(\alpha - \frac{\gamma}{C}\right)\|\nabla u\|_2^2.$$

Applying then Corollary 1 we obtain

$$B(u,u) \geq C_1\|\nabla u\|_2^2 \geq C_2\|u\|_{1,2}^2$$

where C_1 and C_2 are some positive constants. The latter inequality is called also the *coercive* condition.

Now, we notice that, by our assumption, functions $|a_{ji}(x)|$ and $|c(x)|$ are continuous in $\bar{U}$, hence they are bounded there. Denoting by M the common upper bound we find

$$B(u,u) \leq M\ \|u\|_{1,2}^2$$

for all $u \in C_0^\infty(U)$, and, consequently, for all $u \in H_0^{1,2}(U)$.

Summarizing, we conclude that $B(u,v)$ is a symmetric bilinear form satisfying the bilateral inequality

$$m\|u\|_{1,2}^2 \leq B(u,u) \leq M\|u\|_{1,2}^2 \qquad (m, M > 0).$$

Let $|u|_B := \sqrt{B(u,u)}$. Then the new norm is equivalent to $\|u\|_{H^{1,2}}^2$ and we can again apply the Riesz theorem to show that for any bounded linear functional $F(v)$ there is a function $u \in H_0^{1,2}(U)$such that equality (5) above holds.

Thus we have obtained

Theorem. *In the made assumptions, the weak solution of $Lu = 0$ exists and is uniquely defined.*