7. Sobolev inequalities and embedding theorems

The simplest Sobolev imbedding theorem is the following (trivial) inclusion

$$H_0^{1,p}(U) \hookrightarrow L^p(U) \tag{1}$$

which follows immediately from general Poincare-Friedrichs inequality

$$\|v\|_{1,p} \leq C_p \|\nabla v\|_p$$

It turns out that even this information can be made more precise if one takes into account the **dimension** of the ambient space. There two distinguished cases: $p < n$ and $p > n$. The case $p = n$ is also called critical.

We start with the sub-critical case:

Theorem 1(Sobolev inequality: $p < n$) *Let U be a bounded domain in $\mathbb{R}^n$. Then*

$$\|v\|_{p^*} \leq C_p \|\nabla v\|_p , \tag{2}$$

Here

$$p^* = \frac{pn}{n-p}$$

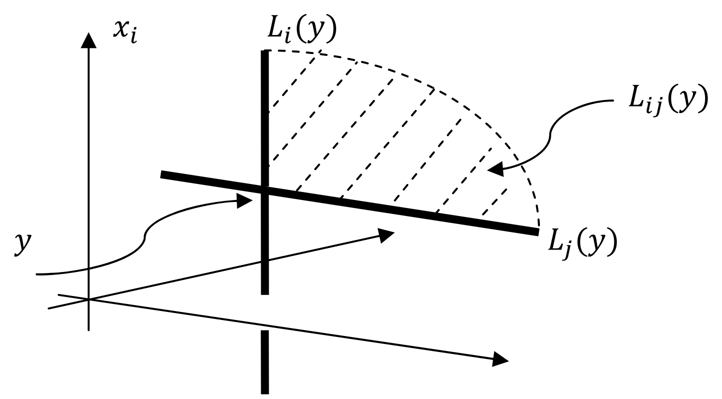
is the so-called ***critical Sobolev's exponent*** *and C_p depends only on p and n.*

■ The crucial step is to prove the Sobolev inequality for

The first case $p = 1$.

Notice that it suffices only to prove (2) for test functions, that is $v \in C_0^\infty(U)$. We extend any given $v \in C_0^\infty(U)$ by zero to the whole $\mathbb{R}^n$ and shall denote for any index $1 \leq i \leq n$ and a point $y \in \mathbb{R}^n$

$$L_i(y) = \{x \in \mathbb{R}^n | x_i = y_i\}, \qquad L_{ij}(y) = \{x \in \mathbb{R}^n | x_i = y_i, x_j = y_j\} \quad \text{etc.}$$



Since $v \in C_0^\infty(U)$ we have for any index i:

$$|v(y)|^{1/(n-1)} = \left|\int_{-\infty}^{y_i} v'_{x_i}(y_1, \dots, x_i, \dots, y_n) dx_i\right|^{1/(n-1)} \le \left(\int_{L_i(y)} |\nabla v|\right)^{1/(n-1)} \equiv h_i^{\frac{1}{n-1}}$$

Here and in what follows in the proof we use the shorthand

$$L_i(y) = L_i, \qquad h_i = \int_{L_i} |\nabla v|, \qquad h_{ij} = \int_{L_i} h_j dx_i \equiv \int_{L_i \oplus L_j} |\nabla v|, \qquad \text{etc.}$$

(the latter integrals should be understood as line, surface integrals with respect to the corresponding measure). Notice also that h_i does not depend on y_i, h_{ij} does not depend on y_i, y_j and so on.

And after multiplication over all $i = 1,2, \dots, n$:

$$|v(y)|^{n/(n-1)} \le h_1^{\frac{1}{n-1}} \cdot h_2^{\frac{1}{n-1}} \cdot \dots \cdot h_2^{\frac{1}{n-1}} \tag{3}$$

Thus, integrating (3) over $L_1(y)$ and applying the Hölder inequality gives

$$\int_{L_1} |v|^{\frac{n}{n-1}} \le h_1^{\frac{1}{n-1}} \int_{L_1} \prod_{i=2}^{n} h_i^{\frac{1}{n-1}} \le h_1^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{L_1} h_i dx_1\right)^{\frac{1}{n-1}} = h_1^{\frac{1}{n-1}} \prod_{i=2}^{n} h_{1i}^{\frac{1}{n-1}}$$

Writing the last product as

$$h_1^{\frac{1}{n-1}} \prod_{i=2}^{n} h_{1i}^{\frac{1}{n-1}} = h_{12}^{\frac{1}{n-1}} \cdot h_1^{\frac{1}{n-1}} \cdot \prod_{i=3}^{n} h_{1i}^{\frac{1}{n-1}}$$

and integrating over $L_2(y)$ (recall that h_{12} does not depend on y_1 and y_2) with application the Hölder inequality yields

$$\int_{L_{12}} |v|^{\frac{n}{n-1}} \equiv \int_{L_2} \int_{L_1} |v|^{\frac{n}{n-1}} \le h_{12}^{\frac{1}{n-1}} \int_{L_2} \left(h_1^{\frac{1}{n-1}} \prod_{i=3}^{n} h_{1i}^{\frac{1}{n-1}}\right) \le h_{12}^{\frac{2}{n-1}} \cdot \prod_{i=3}^{n} h_{12i}^{\frac{1}{n-1}}$$

Applying this argument, we obtain easily by induction that for any $k \le n - 1$

$$\int_{L_{12\dots k}} |v|^{\frac{n}{n-1}} \le h_{12\dots k}^{\frac{k}{n-1}} \cdot \prod_{i=k+1}^{n} h_{12\dots ki}^{\frac{1}{n-1}}$$

Hence for $k = n - 1$ we have

$$\int_{L_{12\dots n-1}} |v|^{\frac{n}{n-1}} \le h_{12\dots n-1} \cdot h_{12\dots n-1,n}^{\frac{1}{n-1}}$$

Integrating this inequality over L_n, and taking into account that $h_{12\dots n-1,n}^{\frac{1}{n-1}}$ does not depend on y_n and that $\mathbb{R}^n = L_{12\dots n}$, we find

$$\int_{\mathbb{R}^n}|v|^{\frac{n}{n-1}} \equiv \int_{L_{12\dots n}}|v|^{\frac{n}{n-1}} \le h_{12\dots n-1,n}^{\frac{1}{n-1}}\int_{L_n} h_{12\dots n-1} = h_{12\dots n}^{\frac{n}{n-1}} \equiv \left(\int_{\mathbb{R}^n}|\nabla v|\right)^{\frac{n}{n-1}}$$

So we have proved the Sobolev inequality for $p = 1$.

The second case: $1 < p < n$.

Now, let us denote by $w = |v|^s$, where $s > 0$ and $v \in C_0^\infty(\mathbb{R}^n)$. It is easy to see that $w \in C_0^\infty(\mathbb{R}^n)$. Hence applying the Sobolev inequality for $p = 1$ to w and then the Hölder inequality, we get

$$\left(\int_{\mathbb{R}^n}|v|^{\frac{sn}{n-1}}\right)^{\frac{n-1}{n}} \equiv \left(\int_{\mathbb{R}^n}|w|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le \int_{\mathbb{R}^n}|\nabla w| = s\int_{\mathbb{R}^n}|v|^{s-1}|\nabla v| \le$$

(by Hölder's inequality)

$$\le s\left(\int_{\mathbb{R}^n}|v|^{\frac{(s-1)p}{p-1}}\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^n}|\nabla v|^p\right)^{\frac{1}{p}}$$

Let us choose s so that

$$\frac{sn}{n-1} = \frac{(s-1)p}{p-1},$$

that is $s = \frac{(n-1)p}{n-p}$. Then we find for this value of s:

$$\left(\int_{\mathbb{R}^n}|v|^{\frac{pn}{n-p}}\right)^{\frac{1}{p}-\frac{1}{n}} \le s\left(\int_{\mathbb{R}^n}|\nabla v|^p\right)^{\frac{1}{p}}$$

which is equivalent to the required inequality

$$\|v\|_{\frac{pn}{n-p}} \le s\,\|\nabla v\|_p$$

The theorem is proved. ■

Corollary 1 (Sobolev embedding theorem for $p < n$). *Let U be a bounded domain in $\mathbb{R}^n$. Then for $p < n$*

$$H_0^{1,p}(U) \hookrightarrow L^q(U), \qquad \text{if } 1 \le p \le q \le p^* \equiv \frac{np}{n-p}. \tag{4}$$

and the embedding continuous in the sense that the following inequality true:

$$\|v\|_q \le C_p\|v\|_{1,p}, \qquad p \le q < p^*.$$

■ Proof: apply the Hölder inequality. ■

In other words, taking into account inequality $p^* \equiv \frac{p}{1-\frac{p}{n}} > p$, we have the following diagram (recall that U is a bounded domain):

$$\ldots \subset C^0(\bar{U}) \subset L^\infty(U) \subset \boldsymbol{H_0^{1,p}}(U) \subset \boxed{\boldsymbol{L^{p^*}}(U) \subset \boldsymbol{L^q}(U) \subset \boldsymbol{L^p}(U)} \subset L^1(U) \qquad (q \geq p).$$

$$\longleftrightarrow$$

$$p^* - p = \frac{p^2}{n-p} \sim \frac{p^2}{n}$$

Finally we prove the super-critical case of the Sobolev inequality.

Theorem 2(Sobolev inequality: $p > n$) *Let U be a bounded domain in $\mathbb{R}^n$. Then*

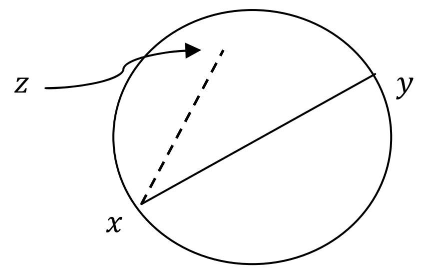
$$H_0^{1,p}(U) \subset C^{0,1-\frac{n}{p}}(\bar{U}), \qquad p > n.$$

Moreover, the embedding $i: H_0^{1,p}(U) \hookrightarrow C^{0,1-\frac{n}{p}}(U)$ is continuous in the sense that the following inequality true:

$$\frac{|v(x) - v(y)|}{|x-y|^{1-\frac{n}{p}}} \leq C(U,n,p)\|\nabla v\|_p$$

*The latter is called **Morrey's inequality**.*

■ We consider $v \in C_0^\infty(U)$ and extend it by zero outside U so that $v \in C_0^\infty(\mathbb{R}^n)$. For any fixed pair of points $x, y \in U$ we denote by B the ball centered at $\frac{x+y}{2}$ of radius $R \equiv \frac{|x-y|}{2}$:



The points of segment $[x,z]$ can be parameterized by: $x + t(z-x)$, when $t \in [0,1]$. We have

$$v(z) - v(x) = \int_0^1 \frac{d}{dt} v\big(x + t(z-x)\big)dt \leq \int_0^1 \big|\nabla v\big(x + t(z-x)\big)\big| \cdot |z-x| dt$$

Integrating the obtained inequality over all points $z \in B$ and dividing by the measure of the ball $|B| = \Omega_n R^n$ (here Ω_n stands for the n-dimensional volume of the n-dimensional unit ball) gives

$$\frac{1}{\Omega_n R^n}\int_B v(z)\,dz - v(x) \le \frac{1}{\Omega_n R^n}\int_B |z-x|\,dz \int_0^1 \left|\nabla v\big(x+t(z-x)\big)\right| \cdot dt$$

We have also $|z-x| \le 2R$ for any z in the ball B. Hence, passing to the absolute values and applying Fubini's theorem we find

$$\left|\frac{1}{\Omega_n R^n}\int_B v(z)\,dz - v(x)\right| \le \frac{2}{\Omega_n R^{n-1}}\int_0^1 dt \int_B \left|\nabla v\big(x+t(z-x)\big)\right| dz \qquad (5)$$

Applying the (linear) change of variables $\xi(z) = x + t(z-x)$ with Jacobian $\frac{dz}{d\xi} = t^{-n}$ we obtain for the inner integral:

$$\int_B \left|\nabla v\big(x+t(z-x)\big)\right| dz = \int_{B'} |\nabla v(\xi)| \frac{dz}{d\xi}\,d\xi = \frac{1}{t^n}\int_{B'} |\nabla v(\xi)|\,d\xi \le$$

by the Hölder inequality

$$\le \frac{1}{t^n}\left(\int_{B'} |\nabla v(\xi)|^p\,d\xi\right)^{\frac{1}{p}} \cdot \left(\int_{B'} 1\,d\xi\right)^{\frac{p-1}{p}} \le \frac{1}{t^n}\|\nabla v\|_p \cdot \left(\Omega_n t^n R^n\right)^{\frac{p-1}{p}}$$

Here we used the fact that $B' = \xi(B)$ is a ball of radius tR. The substitution of the found relations into (5) implies

$$\left|\frac{1}{\Omega_n R^n}\int_B v(z)\,dz - v(x)\right| \le C_1\, R^{1-\frac{n}{p}}\, \|\nabla v\|_p \int_0^1 t^{-\frac{n}{p}} dt$$

Notice that for $p > n$ the integral $\int_0^1 t^{-\frac{n}{p}} dt$ converges, so that we find (recalling that $R = \frac{1}{2}|x-y|$)

$$|a - v(x)| \le C_2\, |x-y|^{1-\frac{n}{p}}\, \|\nabla v\|_p$$

and changing the roles $x \leftrightarrow y$:

$$|a - v(y)| \le C_2\, |x-y|^{1-\frac{n}{p}}\, \|\nabla v\|_p$$

where $a = \frac{1}{\Omega_n R^n}\int_B v(z)\,dz$. Applying the triangle inequality to the last two inequalities we arrive at

$$|v(x) - v(y)| \le |a - v(x)| + |a - v(y)| \le C_3\, |x-y|^{1-\frac{n}{p}}\, \|\nabla v\|_p$$

The theorem is proved. ■