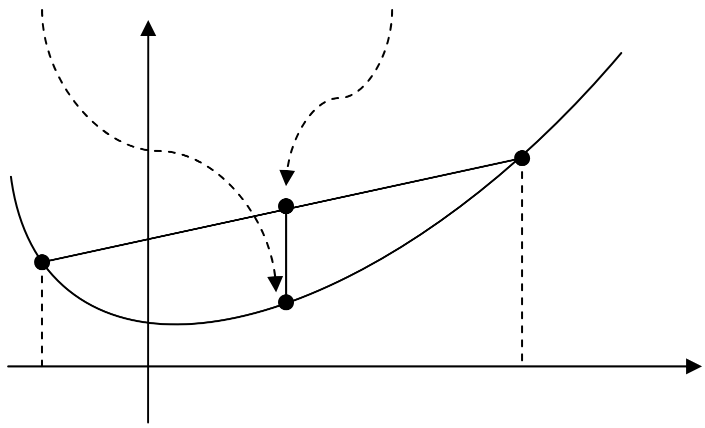
Maximum principles

Some motives:

- *Linear functions* on $[a,b] \subset \mathbb{R}^1$: solutions of differential equation $u''(x) = 0$
- *Convex functions*: if smooth then it is defined as a solution of inequality $u''(x) \geq 0$, otherwise satisfy the structural condition:

$$u\big(xt + y(1-1)\big) \leq xu(x) + (1-t)u(y), \qquad \forall\, t \in [0,1], \qquad \forall x, y \in [a,b]$$



For all these examples the following version of the (**weak**) **maximum principle** holds:

$$\sup_{x\in U} f(x) = \max_{x\in\partial U} f(x)$$

Moreover, for these examples the following (**strong**) **maximum principle** holds:

if $x_0 \in [a,b]$ is an inner point then

- either $f \equiv \text{const}$
- or $f(x_0) < \max_{x\in\partial U} f(x)$

Higher-dimensional analogues

- *linear functions* $\rightarrow$ *harmonic functions* (solutions of $\Delta f(x) = 0$) satisfy the mean-value property
$$\frac{1}{|B_y(R)|}\int_{B_y(R)} f(x)dx = f(y), \qquad \text{where } B_y(R) = \{x \in \mathbb{R}^n : |x-y| < R\}$$

  in particular, the strong maximum principle also holds
- *convex functions* $\rightarrow$ *subharmonic functions* (solutions of $\Delta f(x) \geq 0$) satisfy the mean-value property
$$\frac{1}{|B_y(R)|}\int_{B_y(R)} f(x)dx \geq f(y),$$

The maximum principle in the above sense depends on point-wise properties of a function, that is it requires some regularity the function. For example, it is meaningless for L^p-classes in the above form.

Some historical remarks:

- For harmonic and subharmonic functions was known to Gauss on the basis of the mean value property (1839)
- An extension to elliptic equations and inequalities remained open until the 20th century: S.N. Bernstein (1904) and E. Picard (1905) obtained various results by using analyticity or higher regularity assumptions
- E. Hopf (1927) a transparent and powerful approach which gave an enormous number of applications in many further directions
- Many modifications like comparison (touching) principle are very useful in connection to various geometrical problems (curvature estimates etc.)

We discuss briefly the basic maximum principles for (sub-, super-)solutions of *elliptic* type PDEs, for instance, linear operators having the form

$$Lu = \sum_{i,j=1}^{n} a_{ij}\, u''_{x_i x_j} + \sum_{i=1}^{n} b_i\, u'_{x_i} + cu$$

In this case we always assume that all coefficients are (symmetric) continuous functions: $a_{ij}(x) = a_{ji}(x)$, $a_{ji}(x) \in C^0(\overline{U})$, $b_i(x), c(x) \in C^0(\overline{U})$ and $a_{ij}(x)$ satisfying the (uniform) *ellipticity condition*:

There is a constant $\alpha > 0$ *such that for any* $\xi \in \mathbb{R}^n$

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \geq \alpha \|\xi\|^2.$$

Remark. A function satisfying $Lu \geq 0$ ($Lu \leq 0$ respectively) is called a supersolution (subsolution respectively).

Weak maximum principle

Theorem. *Assume that* $u \in C^2(U) \cap C(\overline{U})$ *and* $c \equiv 0$ *in* U. *If* $Lu \geq 0$ *in* U *then*

$$\max_{x \in \overline{U}} u(x) = \max_{x \in \partial U} u(x).$$

Proof. Let us first suppose that we have a strict inequality $Lu > 0$ in U and yet there exists a point $x_0 \in U$ with $\max_{x \in \overline{U}} u(x) = u(x_0)$. Then x_0 is an inner maximum point, that is we have the following necessary conditions:

$$Du(x_0) = 0$$

and

$$D^2u(x_0) \le 0.$$

Since the matrix $A = \left(a_{ij}(x_0)\right)_{ij=1}^{n}$ is symmetric and positive definite, there exists an orthogonal matrix $\Lambda = (\lambda_{ij})$ such that

$$\Lambda A \Lambda^T = \text{diag}\,(d_1, d_2, \dots, d_n), \qquad d_l > 0.$$

Then we have for any $y = x_0 + \Lambda(x - x_0)$ we have

$$x - x_0 = \Lambda^{-1}(y - x_0) = \Lambda^T(y - x_0)$$

so that

$$u_{x_i} = \sum_{1\le k\le n} u'_{y_k}\lambda_{ik}\,, \qquad u''_{x_i x_j} = \sum_{1\le k,l\le n} u''_{y_k y_l}\lambda_{ik}\lambda_{jl}$$

Hence at the point x_0,

$$\sum_{1\le i,j\le n} a_{ij}u''_{x_i x_j} = \sum_{1\le k,l\le n}\sum_{1\le i,j\le n} u''_{y_k y_l}a_{ij}\lambda_{ik}\lambda_{jl} = \sum_{1\le k\le n} d_k u''_{y_k y_k}$$

(since $\sum_{ij} a_{ij}\lambda_{ik}\lambda_{jl} = d_k\delta_{kl}$). Moreover, we have $d_k > 0$ and $u''_{y_k y_k}(x_0) \le 0$, hence

$$Lu = \sum_{1\le i,j\le n} a_{ij}u''_{x_i x_j} \le 0$$

that contradicts our assumption $Lu > 0$. Thus, the first case is proved.

Now return to the weak inequality $Lu \ge 0$. We introduce an auxiliary function

$$u_t(x) = u(x) + te^{Mx_1}, \qquad x \in U, M > 0.$$

The uniform ellipticity implies $a_{ii}(x) \ge \alpha > 0$.

Therefore,

$$Lu_t = Lu + t\,L(e^{Mx_1}) \ge 0 + t\,e^{Mx_1}(M^2a_{11} + b_1M) \ge Mt\,e^{Mx_1}(M\alpha - \|b_1\|_\infty) > 0$$

in U for M large enough. According to previous step we have for any $t > 0$:

$$\max_{x\in\overline{U}} u_t(x) = \max_{x\in\partial U} u_t(x)$$

Now we let $t \to 0+$, so that we obtain as the limit relation

$$\max_{x\in\overline{U}} u_0(x) = \max_{x\in\partial U} u_0(x)$$

where $u_0 \equiv u$ and the required property follows.

Corollary. (Weak maximum principle for $c \leq 0$) *Assume that $c \leq 0$ and $Lu \geq 0$ in U. Then*

$$\max_{x \in \overline{U}} u(x) \leq \max_{x \in \partial U} u^+(x),$$

where $u^+ = \max\{u, 0\}$. If $Lu = 0$ then

$$\max_{x \in \overline{U}} |u(x)| \leq \max_{x \in \partial U} |u(x)|.$$

Proof. If $u \leq 0$ in U then our statement is trivial. Let us assume now that the set

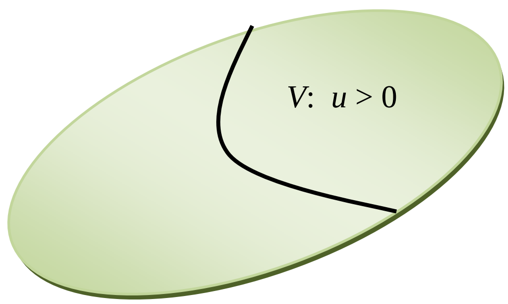
$$V = \{x \in U |\ u(x) > 0\}$$

is non-empty. Then

$$Ku = Lu - cu \geq 0, \qquad x \in V.$$

On the other hand, we can apply the previous theorem to our new operator K since it contains no zero-term (c-term). We have (see picture)

$$\max_{x \in \overline{V}} u(x) = \max_{x \in \partial V} u(x) = \max_{x \in \partial U} u^+(x)$$



In order to prove the second statement one has to apply the previous result to $(-u)$. ■

Strong maximum principle

Theorem (Hopf's lemma, 1927) *Assume that $u \in C^2(U) \cap C^1(\overline{U})$ and $c \equiv 0$ in U. Suppose further that $Lu \geq 0$ in U and that there exists a point $x_0 \in \partial U$ such that*

$$u(x_0) > u(x), \qquad \text{for all } x \in U.$$

Assume also that U is regular at x_0 in the sense that there exists an open ball $B \subset U$ with $x_0 \in \partial B$ (this condition is called **the interior ball condition***). Then*

- $\frac{\partial u}{\partial \nu}(x_0) > 0$, *where ν is the outward unit normal to B at x_0*
- *if $c \leq 0$ then the same conclusion holds provided $u(x_0) \geq 0$.*

Remark. A weak inequality $\frac{\partial u}{\partial \nu}(x_0) > 0$ is trivial and follows immediately from usual maximum property, so that significance of the theorem is the strict inequality $\frac{\partial u}{\partial \nu}(x_0) > 0$.

Corollary (Strong maximum principle). *Let U be a connected open set and $u \in C^2(U) \cap C^1(\overline{U})$ be a solution of $Lu \geq 0$ in U with $c \equiv 0$ in U. If u attains its maximum over $\overline{U}$ at some interior point, then $u \equiv const$ in U.*

Proof (of Corollary**).** Write $M = \max_{\overline{U}} u$ and consider the level set

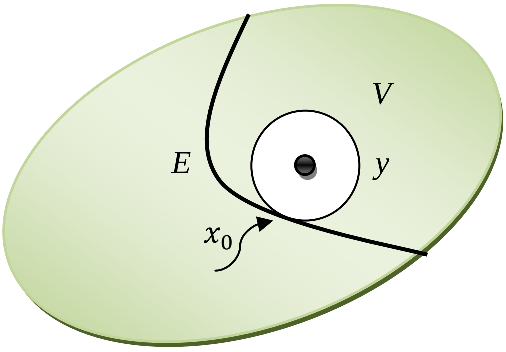
$$E = \{x \in U |\, u(x) = M\}.$$

Then if $u \not\equiv M$ then we can consider the set

$$V = \{x \in U \mid u(x) < M\}.$$

Choose a point $y \in V$ satisfying

$$\text{dist}(y, E) < \text{dist}(y, \partial U)$$



and denote by B the largest ball with center y whose interior lies in V. Then there exists some point $x_0 \in E \cap \partial B$. Clearly, V satisfies the interior ball condition at x_0, whence Hopf's Lemma implies $\frac{\partial u}{\partial \nu}(x_0) > 0$. But This is a contradiction since u attains its maximum at x_0 (that is we have $Du(x_0) = 0$).

Proof of Hopf's Lemma

Assume first that $c \leq 0$ and $u(x_0) \geq 0$. Without loss of generality, $B = B_0(r)$, for some radius $r > 0$. Define

$$v(x) = e^{-\lambda|x|^2} - e^{-\lambda r^2}, \qquad x \in B_0(R), \qquad \lambda > 0.$$

Then using the uniform ellipticity condition, we find

$$Lv = \sum_{i,j=1}^{n} a_{ij}\, v''_{x_i x_j} + \sum_{i=1}^{n} b_i\, v'_{x_i} + cu$$

$$= e^{-\lambda|x|^2} \sum_{i,j=1}^{n} a_{ij} \left(4\lambda^2 x_i\, x_j - 2\lambda\delta_{ij}\right) - 2\lambda e^{-\lambda|x|^2} \sum_{i=1}^{n} b_i\, x_i + c\left(e^{-\lambda|x|^2} - e^{-\lambda r^2}\right)$$

$$\geq e^{-\lambda|x|^2}(4\alpha\lambda^2|x|^2 - 2\lambda \operatorname{tr}(A) - 2\lambda b \cdot |x| + c)$$

(recall that $c \leq 0$, hence $-ce^{-\lambda r^2} \geq 0$), where $b = \sup_{x\in\bar{U}} \sqrt{b_1^2 + \cdots + b_n^2}$.

Consider the open annular region $B_0(r) \setminus B_0\left(\frac{r}{2}\right)$. We have

$$Lv \geq e^{-\lambda|x|^2}(\alpha^2\lambda^2 r^2 - 2\lambda \operatorname{tr}(A) - 2\lambda br + c) \geq 0$$

if λ is chosen large enough. Hence $Lv \geq 0$ in $B_0(r) \setminus B_0\left(\frac{r}{2}\right)$.

In view of our condition $u(x_0) > u(x)$ we can find a constant $\epsilon > 0$ such that

$$u(x_0) \geq u(x) + \epsilon\, v(x), \qquad x \in \partial B_0\left(\frac{r}{2}\right). \qquad (*)$$

In addition we have (since $v \equiv 0$ on $\partial B_0(r)$)

$$u(x_0) \geq u(x) + \epsilon\, v(x), \qquad x \in \partial B_0(r) \qquad (**)$$

Thus, applying $Lu \geq 0$ and $Lv \geq 0$, we see

$$L\big(u + \epsilon v - u(x_0)\big) \geq -cu(x_0) \geq 0, \qquad x \in B_0(r) \setminus B_0\left(\frac{r}{2}\right).$$

From (*) and (**) we have

$$u(x) + \epsilon\, v(x) - u(x_0) \leq 0 \qquad \text{on} \qquad \partial\left(B_0(r) \setminus B_0\left(\frac{r}{2}\right)\right)$$

By virtue of weak maximum principle we obtain that that latter inequality holds everywhere in $B_0(r) \setminus B_0\left(\frac{r}{2}\right)$. But

$$u(x_0) + \epsilon\, v(x_0) - u(x_0) = 0$$

and so we have the normal derivative

$$0 \leq \partial_\nu\big(\, u(x) + \epsilon\, v(x) - u(x_0)\big)|_{x_0} = \partial_\nu u(x_0) + \epsilon\, \partial_\nu v(x_0)$$

It follows that

$$\partial_\nu u(x_0) \geq -\epsilon\partial_\nu v(x_0) = -\frac{\epsilon}{r} Dv(x_0) \cdot x_0 = 2\lambda\epsilon r\, e^{-r^2} > 0.$$

■