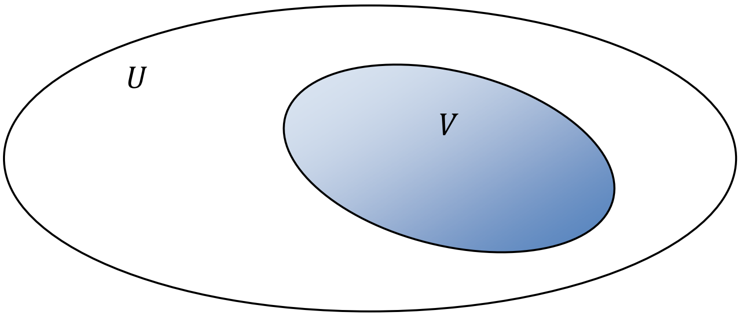

Nomenclature:

$$V \subset\subset U$$

$$d(V, \partial U) := \mathrm{dist}\,(V, \partial U)$$

$$D_j^h u(x) = \frac{u\left(x + he_j\right) - u(x)}{h}, \qquad x \in V, \qquad |h| < d(V, \partial U).$$

$$D^h u = (D_1^h u, \dots, D_n^h u)$$

- in general, $D_j^h u(x)$ is well defined almost everywhere in V
- D_j^h is a linear operator: $L^1(V) \to L^1(U)$
- D_j^h can be thought of as an approximation of partial derivative ∂_{x_j}

Integration by parts: For any $\varphi \in C_0^\infty(V)$ and all h, $0 < |h| < d(\mathrm{supp}\,\varphi, \partial U)$

$$\int_U u\, D_i^h \varphi\, dx \; = -\int_U \varphi\, D_i^{-h} u\, dx$$

■(“Extended version” of Proof). Denote by $V = \mathrm{supp}\,\varphi$. Then

$$\begin{aligned}
\int_U u\, D_i^h \varphi\, dx \; &= \frac{1}{h}\int_U u(x)\varphi\left(x + he_j\right)dx - \frac{1}{h}\int_U u(x)\varphi(x)dx \\
&= \frac{1}{h}\int_{\boldsymbol{V - he_j}} \boldsymbol{u(x)\varphi\left(x + he_j\right)dx} - \frac{1}{h}\int_V u(x)\varphi(x)dx \\
&= \frac{1}{h}\int_{\boldsymbol{V}} \boldsymbol{u\left(y + he_j\right)\varphi(y)dy} - \frac{1}{h}\int_V u(x)\varphi(x)dx \\
&= \int_{\boldsymbol{V}} \varphi(x)\left(\frac{u\left(x - he_j\right) - u(x)}{h}\right)dx \\
&= -\int_U \varphi\, D_i^{-h} u\, dx \;. \blacksquare
\end{aligned}$$

Leibniz rule. For any two admissible functions the following (if well-defined) holds

$$D_k^h(uv) = u^h D_k^h(v) + D_k^h(u)v,$$

Here $u^h(x) = u(x + he_k)$.

Lemma 1. *If* $1 \le p < \infty$ *and* $u \in H^{1,p}(U)$ *then for each* $V \subset\subset U$

$$\|D^h u\|_{L^p(V)} \le C\, \|Du\|_{L^p(U)}$$

for some constant $C = C(n,p)$ *and all* $0 < |h| < \text{dist}\,(V, \partial U)$.

Proof (Extended version).

Assume first that u is smooth. Then for any $x \in V$ and $0 < |h| < \text{dist}\,(V, \partial U)$

$$u(x + he_i) - u(x) = h \int_0^1 u'_{x_i}(x + the_i)\, dt$$

so that

$$\left|D_i^h u(x)\right| = \frac{|u(x + he_i) - u(x)|}{|h|} \le \int_0^1 |Du(x + the_i)|\, dt$$

Then

$$|D^h u(x)|^p = \left(\sum_{i=1}^n \left|D_i^h u(x)\right|^2\right)^{\frac{p}{2}} \le n^{\frac{p}{2}} \left(\max_{1 \le i \le n} \left|D_i^h u(x)\right|\right)^p \le n^{\frac{p}{2}} \sum_{i=1}^n \left|D_i^h u(x)\right|^p$$

$$\le n^{\frac{p}{2}} \sum_{i=1}^n \left(\int_0^1 |Du(x + the_i)|\, dt\right)^p \le$$

(by Hölder)

$$\le n^{\frac{p}{2}} \sum_{i=1}^n \int_0^1 |Du(x + the_i)|^p\, dt$$

Integration over V:

$$\int_V |D^h u(x)|^p\, dx \le n^{p/2} \sum_{i=1}^n \int_V dx \int_0^1 |Du(x + the_i)|^p\, dt = n^{p/2} \sum_{i=1}^n \int_0^1 dt \int_V |Du(x + the_i)|^p dx$$

Thus

$$\int_V |D^h u|^p\, dx \le n^{p/2} \int_U |Du|^p\, dx$$

This estimate holds for smooth u, hence it is valid by approximation for arbitrary $u \in H^{1,p}(U)$ and the lemma follows for $C = n^{p/2}$.

∎

Lemma 2. *If $u \in L^p(U)$ for $1 < p < \infty$ and for some $V \subset\subset U$ there exists a constant C such that for all $0 < |h| < \operatorname{dist}(V, \partial U)$ the inequality holds*

$$\|D^h u\|_{L^p(V)} \leq C$$

Then $u \in H^{1,p}(V)$ and

$$\|Du\|_{L^p(V)} \leq C_1 = C_1(n, p, C).$$

Proof. Consider $h_j = \frac{d}{j}$, where $d = \frac{1}{2}\operatorname{dist}(V, \partial U)$.

- $\left\|D^{h_j} u\right\|_{L^p(V)} \leq C \quad \Rightarrow \quad$ for any fixed $i = 1, 2, \dots, n$, the family $\left\{D_i^{h_j} u\right\}_{j=1,2,\dots}$ is bounded in $L^p(V)$-sense.
- In a reflexive Banach space (that is $X^{**} = X$) any bounded set is weakly compact
- Since $p > 1 \quad \Rightarrow \quad L^{p^*} = L^q \left(\frac{1}{p} + \frac{1}{q} = 1\right) \quad \Rightarrow \quad L^p$-space is a reflexive Banach space, hence a bounded sequence in $L^p(V)$ is weakly compact
- Thus (for any index $i \leq n$) we can find a subsequence $h_k \to 0$ ($h_k \equiv h_{j_k}$) and a function $v_i \in L^p(V)$ such that

$$D_i^{h_k} u \xrightarrow{\text{weakly in } L^p(V)} v_i.$$

- This, in its turn, means that for any $g \in L^q(U) \equiv L^p(V)^*$ we have

$$\lim_{k\to\infty} \int_V g\, D_i^{h_k} u\, dx = \int_V g v_i\, dx\,.$$

- Since $C_0^\infty(V) \subset L^q(U)$, for any $\varphi \in C_0^\infty(V)$

$$\lim_{k\to\infty} \int_V \varphi\, D_i^{h_k} u\, dx = \int_V \varphi\, v_i\, dx$$

- Integration by parts (in the version for difference quotients) yields

$$\int_U \varphi\, D_i^{h_k} u\, dx = -\int_U u\, D_i^{-h_k} \varphi\, dx$$

- Thus

$$\int_V \varphi v_i\, dx = \lim_{k\to\infty} \int_V \varphi\, D_i^{h_k} u\, dx = -\lim_{k\to\infty} \int_U u\, D_i^{-h_k} \varphi\, dx$$

- But $D_i^{-h_k} \varphi \rightrightarrows \varphi_{x_i}$ (converges uniformly in U as $k \to \infty$), hence

$$\lim_{k\to\infty} \int_U u\, D_i^{-h_k} \varphi\, dx = \int_U u \varphi_{x_i}\, dx = \int_V u \varphi_{x_i}\, dx$$

- It follows that
$$\int_V u\varphi_{x_i}\,dx = -\int_V \varphi v_i\,dx$$
- This proves that v_i is the weak derivative of u with respect to x_i in V, in particular, $v_i \in L^p(V)$
- Thus the weak gradient $Du \in L^p(V)$. Since u is itself in $L^p(V)$, we deduce therefore that $u \in H^{1,p}(V)$.
- Finally, setting $g = v_i^{\frac{p}{q}} \in L^q(V)$ in
$$\lim_{k\to\infty}\int_V g\,D_i^{h_k}u\,dx = \int_V g v_i\,dx$$
we obtain (notice that $1+\frac{p}{q} = p\left(\frac{1}{p}+\frac{1}{q}\right) = p$)
$$\int_V v_i^{\frac{p}{q}} v_i\,dx \equiv \int_V v_i^p\,dx = \lim_{k\to\infty}\int_V v_i^{\frac{p}{q}}\,D_i^{h_k}u\,dx \le$$
$$\le \left(\int_V v_i^p\,dx\right)^{\frac{1}{q}} \limsup_{k\to\infty}\left(\int_V \left|D_i^{h_k}u\right|^p dx\right)^{\frac{1}{p}}$$
$$\le C^{1/p}\left(\int_V v_i^p\,dx\right)^{\frac{1}{q}}$$
Hence, dividing we obtain
$$\int_V v_i^p\,dx \le C$$
which yields the required estimate of the weak gradient:
$$\|Du\|_{L^p(V)} \le C_1(n,p)C$$

The lemma is proved completely. ■

Inner regularity for $Lu \equiv -\sum_{i,j=1}^{n} a_{ij}\, u''_{x_i x_j} + \sum_{i=1}^{n} b_i\, u'_{x_i} + cu = f$

Structure conditions: $a_{ij}(x) = a_{ji}(x)$ and

$$a_{ji}(x) \in C^1(U), \qquad b_i(x), c(x) \in L^\infty(U), \qquad f \in L^2(U).$$

The uniform ellipticity: *there is a constant* $\alpha > 0$ *such that for any* $\xi \in \mathbb{R}^n$ *and all* $x \in U$

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \geq \alpha \|\xi\|^2.$$

Main Theorem. *Let* $u \in H^{1,2}(U)$ *be a weak solution of* $Lu = f$.

Then $u \in H^{2,2}_{loc}(U)$ *and for any open subset* $V \subset\subset U$ *the following estimate holds:*

$$\|u\|_{H^{2,2}(V)} \leq C\big(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\big).$$

Here $C = C(L, U, V)$.

Corollary. *We have the classical identity*

$$Lu = f \qquad \text{a.e. in } U$$

Indeed, for any function $v \in C_0^\infty(U)$ we have the weak identity

$$B(u, v) = (f, v).$$

On the other hand, since $u \in H^{2,2}_{loc}(U)$ we can integrate by parts the left hand side:

$$B(u, v) = (Lu, v).$$

Thus $(Lu - f, v) = 0$ for all $v \in C_0^\infty(U)$, and so $Lu = f$ a.e. in U.

Proof of the Main Theorem. We choose arbitrarily some open set W such that

$$V \subset\subset W \subset\subset U$$

and select some smooth function $\zeta(x)$ satisfying

$$0 \leq \zeta \leq 1, \qquad \zeta = 1 \text{ on } V \qquad \text{and} \quad \zeta = 0 \text{ on } \mathbb{R}^n \setminus W.$$

Such a function is called a *cutoff function*.

Since u is a weak solution we have $B(u, v) = (f, v)$ for all $v \in H_0^{1,2}(U)$, hence

$$\sum_{i,j=1}^{n} \int_U a_{ij} u'_{x_i} v'_{x_j}\, dx = \int_U \tilde{f}\, v\, dx, \qquad \tilde{f} \coloneqq f - \sum_i^n b_i u_{x_i} - cu \in L^2(U) \tag{$*$}$$

- For any an index $k \leq n$ consider the test function

$$v = -D_k^{-h}(\zeta^2 D_k^h u)$$

- $v \in H_0^{1,2}(U)$ for small h
- We have

$$A := \sum_{i,j=1}^{n} \int_U a_{ij} u'_{x_i} v'_{x_j}\, dx = \int_U \tilde{f}\, v\, dx =: B$$

1) *Estimate of A.* Integration by parts and Leibniz rule (in the difference quotients versions) yield:

$$A = -\sum_{i,j=1}^{n} \int_U a_{ij} u'_{x_i} \left(D_k^{-h}\left(\zeta^2 D_k^h u\right)\right)'_{x_j} dx =$$

$$= \sum_{i,j=1}^{n} \int_U D_k^h\left(a_{ij} u'_{x_i}\right)\left(\zeta^2 D_k^h u\right)'_{x_j} dx =$$

$$= \sum_{i,j=1}^{n} \int_U a_{ij}^h D_k^h\left(u'_{x_i}\right)\left(\zeta^2 D_k^h u\right)'_{x_j} dx + \int_U u_{x_i} D_k^h\left(a_{ij}\right)\left(\zeta^2 D_k^h u\right)'_{x_j}$$

Here $a_{ij}^h(x) = a_{ij}(x + h e_k)$. Expanding the derivative $\left(\zeta^2 D_k^h u\right)'_{x_j}$ we obtain

$$A = A_1 + A_2$$

where

$$A_1 = \sum_{i,j=1}^{n} \int_U a_{ij}^h \zeta^2 D_k^h\left(u'_{x_i}\right)\ D_k^h(u'_{x_j}) dx$$

and

$$A_2 = \sum_{i,j=1}^{n} \int_U \left[2\zeta\zeta'_{x_j} a_{ij}^h D_k^h\left(u'_{x_i}\right) D_k^h(u) \ + \zeta^2 D_k^h\left(a_{ij}\right) u_{x_i} D_k^h\left(u'_{x_j}\right) + 2\zeta\zeta'_{x_j} u_{x_i} D_k^h\left(a_{ij}\right) D_k^h(u)\right]$$

By uniform ellipticity,

$$A_1 \geq \alpha \int_U \zeta^2 \sum_{i}^{n} \left(D_k^h\left(u'_{x_j}\right)\right)^2 dx \ \equiv \alpha \int_U \zeta^2\, |D_k^h(Du)|^2 dx$$

Moreover, we have for some appropriate $C = C(a_{ij}, b_i, c, \zeta') > 0$ from boundedness of the coefficients of L:

$$|A_2| \leq C \int_U \zeta\left(\left|D_k^h Du\right| \cdot \left|D_k^h u\right| + \left|D_k^h Du\right| \cdot |Du| + \left|D_k^h u\right| \cdot |Du|\right) dx$$

Applying Cauchy's inequalities

$$C\zeta\left|D_k^h Du\right| \cdot \left|D_k^h u\right| \leq \frac{\epsilon}{2}\,\zeta^2\left|D_k^h Du\right|^2 + \frac{C^2}{2\epsilon}\left|D_k^h u\right|^2,$$

$$C\zeta\left|D_k^h Du\right| \cdot |Du| \leq \frac{\epsilon}{2}\,\zeta^2\left|D_k^h Du\right|^2 + \frac{C^2}{2\epsilon}|Du|^2,$$

$$C\zeta\left|D_k^h u\right| \cdot |Du| \leq \frac{C\zeta}{2}\,\left|D_k^h u\right|^2 + \frac{C\zeta}{2}|Du|^2$$

Notice that

$$\text{if } \epsilon < C \quad \text{then} \qquad \frac{C}{2} < \frac{C^2}{2\epsilon}.$$

Thus, taking into account that $\zeta \leq 1$ everywhere and $\zeta = 0$ in $\mathbb{R}^n \setminus W$ we get

$$|A_2| \leq \epsilon\int_W \zeta^2\left|D_k^h Du\right|^2 dx + \frac{C^2}{\epsilon}\int_W \left(\left|D_k^h u\right|^2 + |Du|^2\right) dx$$

Now choose additionally $\epsilon < \frac{\alpha}{2}$ and apply the integral estimate for difference quotients (Lemma 1 above)

$$\int_W \left|D_k^h u\right|^2 dx \leq C_1\int_U |Du|^2 dx\ , \qquad C_1 = C_1(n,p).$$

Thus we arrive at

$$|A_2| \leq \frac{\alpha}{2}\int_W \zeta^2\left|D_k^h Du\right|^2 dx + C_2\int_U |Du|^2 dx\ , \qquad C_2 = \frac{C^2}{\epsilon}(1 + C_1).$$

These two estimates for A_k imply together

$$A \geq \frac{\alpha}{2}\int_W \zeta^2\left|D_k^h Du\right|^2 dx - C_2\int_U |Du|^2 dx\,.$$

2) *Estimate of B.* Applying Lemma 1 to v we find

$$\int_U v^2 dx = \int_W v^2 dx \equiv \left\|D_k^{-h}\left(\zeta^2 D_k^h u\right)\right\|_{L^2(W)}^2 \leq C_1\int_U \left|D\left(\zeta^2 D_k^h u\right)\right|^2 dx$$

Here $D_{x_k}\left(\zeta^2 D_k^h u\right) = 2\zeta\zeta'_{x_k} D_k^h u + \zeta^2\left(D_k^h u\right)'_{x_k}$, hence for some $C_3 = C_3(\zeta)$

$$\left|D\left(\zeta^2 D_k^h u\right)\right|^2 \leq C_3\left(\left|D_k^h u\right|^2 + \zeta^2\left|D_k^h Du\right|^2\right)$$

and integration together with application of Lemma 1 to $D_k^h u$ gives

$$\int_U v^2 dx \leq C_4\int_W \left(\left|D_k^h u\right|^2 + \zeta^2\left|D_k^h Du\right|^2\right) dx \leq C_5\int_W \left(|Du|^2 + \zeta^2\left|D_k^h Du\right|^2\right) dx,$$

Appling again the above trick with Cauchy's inequality we find

$$|B| \le C \int_U (|f| + |Du| + |u|)|v|\, dx \le$$

$$\le \epsilon \int_U |v|^2 dx + \frac{C_6}{\epsilon} \int_U (|f|^2 + |Du|^2 + |u|^2) \le$$

$$\le C_5 \epsilon \int_W \zeta^2 \left|D_k^h Du\right|^2 + C_7 \int_U (|f|^2 + |Du|^2 + |u|^2)$$

For $\epsilon = \frac{\alpha}{4C_5}$ this finally yields

$$|B| \le \frac{\alpha}{4} \int_U \zeta^2 \left|D_k^h Du\right|^2 + C_7 \int_U (|f|^2 + |Du|^2 + |u|^2)$$

Recalling that $A = B$ we obtain

$$\frac{\alpha}{4} \int_U \zeta^2 \left|D_k^h Du\right|^2 \le C_8 \int_U (|f|^2 + |Du|^2 + |u|^2)$$

The first integral may be estimated from below since $\zeta = 1$ on V:

$$\int_V \left|D_k^h Du\right|^2 \le C_9\left(\|f\|^2_{L^2(U)} + \|u\|^2_{L^2(U)} + \|Du\|^2_{L^2(U)}\right) = C_9\left(\|f\|^2_{L^2(U)} + \|u\|^2_{H^{1,2}(U)}\right)$$

This uniform bound for $\left\|D_k^h Du\right\|_{L^2(V)}$ by Lemma 2 implies regularity of Du:

$$Du \in H^{1,2}(V) \qquad \Rightarrow \qquad Du \in H^{1,2}_{loc}(U),$$

and therefore $u \in H^{2,2}_{loc}(U)$ and

$$\|u\|^2_{H^{2,2}(V)} \equiv \|u\|^2_{L^2(V)} + \|Du\|^2_{L^2(V)} + \|D^2 u\|^2_{L^2(V)} \le C(L, V, U) \cdot \left(\|f\|^2_{L^2(U)} + \|u\|^2_{H^{1,2}(U)}\right)$$

3) In order to achieve the required L^2-norm of u in the latter estimate we notice that the above argument also yields

$$\|u\|_{H^{2,2}(V)} \le C_1(L, V, W) \cdot \left(\|f\|_{L^2(W)} + \|u\|_{H^{1,2}(W)}\right)$$

By choosing a new cutoff function

$$v = \zeta^2 u, \qquad \zeta = 1 \text{ on } W$$

and applying formula (*) together the uniform ellipticity, one can prove that

$$\int_U \zeta^2 \, |Du|^2 \, dx \le C \int_U (f^2 + u^2)\, dx \qquad (**)$$

Indeed,

$$\sum_{i,j=1}^{n} \int_U a_{ij} u'_{x_i} (\zeta^2 u)'_{x_j} \, dx = \int_U \tilde{f} \, \zeta^2 u \, dx$$

Here

$$\sum_{i,j \le n} a_{ij} u'_{x_i} (\zeta^2 u)'_{x_j} = \sum_{i,j \le n} 2\zeta \zeta'_{x_i} a_{ij} u u'_{x_j} + \zeta^2 a_{ij} u'_{x_i} u'_{x_j} \ge \alpha \zeta^2 |Du|^2 + \sum_{i,j \le n} 2\zeta \zeta'_{x_i} a_{ij} u u'_{x_j}$$

This yields

$$\alpha \int_U \zeta^2 |Du|^2 \, dx \le \int_U \tilde{f} \, \zeta^2 u \, dx - \int_U \sum_{i,j \le n} 2\zeta \zeta'_{x_i} a_{ij} u u'_{x_j} \, dx$$

We have as above

$$\tilde{f} \zeta^2 u = \left(f - \sum_{i}^{n} b_i u_{x_i} - cu \right) \zeta^2 u \le \epsilon \zeta^2 |Du|^2 + C_{10}(\epsilon)(f^2 + u^2)$$

and

$$\sum_{i,j \le n} 2\zeta \zeta'_{x_i} a_{ij} u u'_{x_j} \le \epsilon \zeta^2 |Du|^2 + C_{11}(\epsilon) u^2$$

Combining these estimates we obtain the required inequality. It implies finally that

$$\|u\|_{H^{1,2}(W)} \le C(L, V, W) \cdot \left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)} \right)$$

The theorem is proved completely. ■