Glossary of some classical applications to elliptic equations

Maximum principles

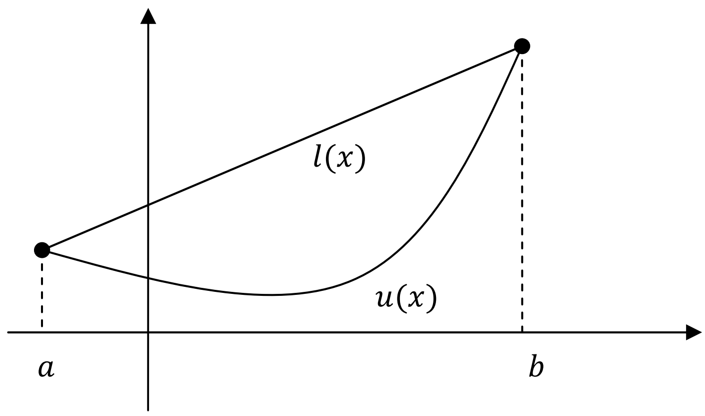
- Carl Friedrich Gauss (1777-1855)
- ...

"if u is a solution of some elliptic inequality $Lu \geq 0$ then it takes its **maximum** value on the boundary"

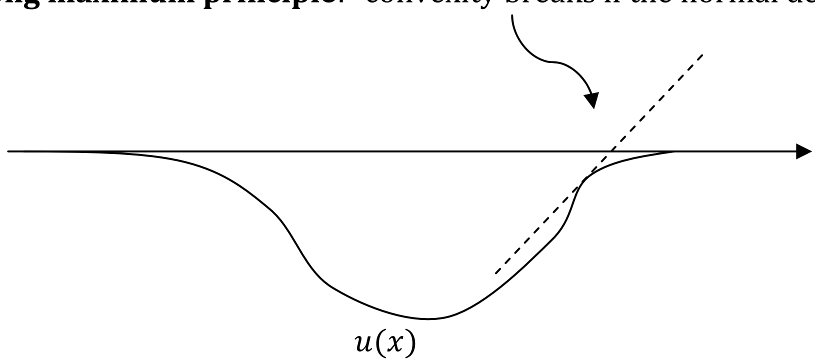
A model example: $Lu = u''(x) \geq 0$.

- Then u is a convex function, hence its behavior "prescribed" by boundary values
- "Barriers": compare u with the linear function $l(x)$ (a **solution** to $Lu = 0$), which solves the Dirichlet problem $u(a) = l(a)$ and $u(b) = l(b)$:

$$u(x) - l(x) \leq \max\{u(a) - l(a), u(b) - l(b)\} = 0, \qquad \forall x \in [a, b]$$



- **Strong maximum principle**: "convexity breaks if the normal derivative is zero"



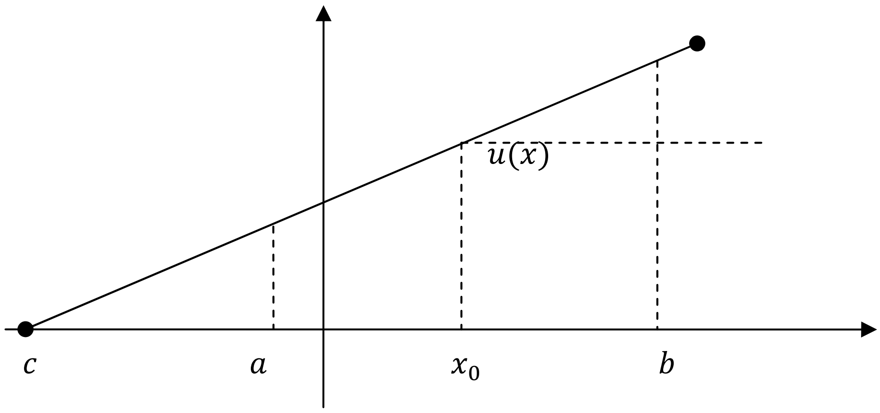
- **Comparison principles**: Laplacian is an invariant operator, can be defined in any Riemannian category $\Rightarrow$ geometric applications (curvature estimates)

Harnack's inequality

- Carl Gustav Axel Harnack (1851-1888), a German mathematician

"if u is a **positive** solution of some elliptic inequality $Lu = 0$ in the ball $B_{x_0}(R)$ then there is a constant $C = C(V)$ such that $\max_U u \le C\, u(x_0)$"

A model example: $Lu = u''(x) = 0$.



$U = [a,b]$: Hernack's inequality follows from **Thales' Theorem**

$$\frac{u(b)}{u(x_0)} = \frac{b-c}{x_0-c} = 1 + \frac{b-x_0}{x_0-c} \le 1 + \frac{b-x_0}{x_0-a} \equiv C$$

Liouville theorems

Joseph Liouville (1809 – 1882), a French mathematician.

"if u is a solution of some elliptic equation $Lu = 0$ and u is **bounded** (from below) then u is identically constant"

- Depend on **dimensions** and structural conditions
- Can be obtained from Harnack's inequality
- Can be proved via energy-capacity estimates

A model of a proof by using capacity.

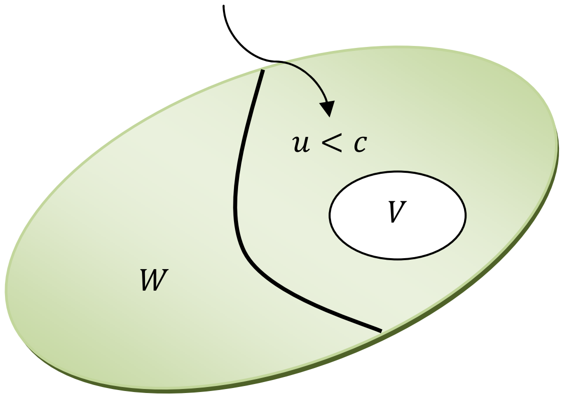
Liouville Theorem for harmonic functions. If u is an entire bounded solution to the Laplace equation:

$$Lu = u''_{x_1x_1} + u''_{x_2x_2} = 0, \qquad u \ge 0 \text{ in } \mathbb{R}^2$$

then $u \equiv const$.

Proof. Let us assume that $u \neq const$ and consider some $x_0 \in \mathbb{R}^2$. Take $c > u(x_0)$ and denote

$$U = \{x \in \mathbb{R}^2 : u(x) < c\} \quad \Rightarrow \quad x_0 \in U \quad \Rightarrow \quad U \neq \emptyset$$



Consider an arbitrary $V \subset\subset U$ and a Lipschitz function $\zeta(x)$ with compact support:

$$\zeta(x) = 1 \text{ on } V \quad \text{and} \quad \zeta = 0 \text{ in } \mathbb{R}^2 \setminus W,$$

where $W = \operatorname{supp} \zeta$. Then

$$g(x) \equiv (u(x) - c)\zeta^2(x)$$

vanishes on the boundary of $U \cap W$. Integrating by parts yields

$$0 = \int_U g\Delta u = -\int_W Dg \cdot Du = -\int_W \zeta^2 |Du|^2 - 2\int_W (u-c)\,\zeta\, D\zeta \cdot Du$$

By Cauchy's inequality (taking into account that $-c \leq u - c \leq 0$ on U)

$$\int_W \zeta^2 |Du|^2 \leq 2c \left(\int_W \zeta^2 |Du|^2\right)^{\frac{1}{2}} \left(\int_W |D\zeta|^2\right)^{\frac{1}{2}}$$

Hence

$$\int_W \zeta^2 |Du|^2 \leq 4c^2 \int_W |D\zeta|^2$$

Take infimum in the right hand side over all admissible ζ and notice that $\zeta = 1$ on V:

$$\int_V |Du|^2 \leq 4c^2 \inf_\zeta \int_{\mathbb{R}^2} |D\zeta|^2 \equiv 4c^2 \operatorname{cap}(V)$$

where the latter variational quantity is the called variational L^2-capacity:

$$\operatorname{cap}(V) = \inf_\zeta \int_{\mathbb{R}^2} |D\zeta|^2$$

In our case, it can be shown that $\operatorname{cap}(V) = 0$ for any compact set. Indeed, let $V = B_0(r)$ and consider the test function

$$\zeta = \frac{\ln \frac{|x|}{R}}{\ln \frac{r}{R}}, \qquad x \in B_0(R) \setminus B_0(r)$$

Then

$$\operatorname{cap}(B_0(r)) = \inf_{\zeta} \int_{\mathbb{R}^2} |D\zeta|^2 \le \frac{1}{\left(\ln \frac{r}{R}\right)^2} \int_0^{2\pi} d\theta \int_r^R \rho \cdot \frac{1}{\rho^2}\, d\rho = \frac{2\pi}{\ln \frac{r}{R}}$$

It follows that

$$\operatorname{cap}(B_0(r)) = 0, \qquad \forall r > 0$$

In particular, the capacity of any compact set is equal to zero.

This proves finally that

$$\int_V |Du|^2 = 0 \quad \Rightarrow \quad u = const \text{ on } V \quad \Rightarrow \quad \text{contradiction! } \blacksquare$$

Remark 1. In fact we used only a weak formulation of our problem, that is one use the capacity methods (and this is more natural) for weak solutions of elliptic equations.

Remark 2. There is a reserve in the above argument: instead of the integral estimate for $|Du|^2$ one can use some other (positive) function $\theta(|Du|)$ which naturally leads to a wider class of quasilinear equations.

Remark 3. The above method is easily generalized on L^p-norms etc.

Phragmén-Lindelöf principle

Lars Edvard Phragmén (1863-1937), a Swedish mathematician

Ernst Leonard Lindelöf (1870-1944), a finnish mathematician

"Either u is **bounded** in a neighborhood of a boundary point or it growth **rapidly**"

Admits also an interpretation in terms of energy-capacity estimates. Namely, in the above proof one can refine the argument to obtain

$$\int_{B_0(r)} |Du|^2 \le 4 \max_{x \in B_0(R)} |u - c|^2 \operatorname{cap}\big(B_0(r), B_0(R)\big) \sim \frac{M^2(r)}{\ln \frac{R}{r}},$$

where $M(R) = \max_{x \in B_0(R)} |u(x) - c|^2$. Hence we have the alternative: either $u \equiv const$ or it growth like $\sqrt{\ln \frac{R}{r}}$.

Denjoy-Ahlfors theorems

Arnaud Denjoy (1884 – 1974), a French mathematician

Lars Valerian Ahlfors (1907 – 1996), a Finnish mathematician

The order of growth of an entire holomorphic function $f(z)$ gives the upper bound on the number N of different asymptotic values of f:

$$\lim_{r\to\infty}\sup\frac{\ln\ln M(R)}{\ln R}\geq\frac{N}{2},\qquad M(R)=\max_{|z|=R}|f(z)|$$

- A key property for generalizations is that $u=|f(z)|$ is a subharmonic function.
- Denjoy-Ahlfors type theorems describe the topological structure of subharmonic functions: how many "caps" one can cut off of the graph of a subharmonic function
- Give a quantative version of Liouville's theorem (by Morse theory, there are at least two "caps")

Bernstein type theorems

Sergei Natanovich Bernstein (1880-1968), a Russian mathematician

"if u is an **entire** (=defined in the whole $\mathbb{R}^n$) solution of some elliptic quasilinear equation $Lu=0$ then it is "trivial" = contains in some small class (constant, linear, polynomial, …)"

- Initially proved for minimal surface equation in $\mathbb{R}^2$

$$\operatorname{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}=0$$

- Bombieri and Giusti: for $n\leq 7$ any entire solution is a linear function; J. Simons proved that for $n\geq 8$ there non-trivial solutions.

- Many examples and similar phenomena, for instance, the well-known De Giorgi conjecture: let u be an **entire** solution of

$$\Delta u=u^3-u,\qquad u\leq 1,\qquad x\in\mathbb{R}^n$$

  satisfying $u'_{x_1}>0$ everywhere. Then for $n\leq 8$ level-sets must be hyperplanes.