Partiella differentialekvationer (D): Exercise problems (2008-10-07)

1. Prove the Leibniz rule for distributions: if $F \in \mathcal{D}'(U)$ and $f \in C^\infty(U)$, where $U \in \mathbb{R}^n$ then
$$D(f \cdot F) = D(f) \cdot F + f \cdot D(F)$$
for $D = \partial_{x_i}$. (Hint: apply the both sides to a test function)

2. Let $F(\varphi) = \sum_{k=1}^{\infty} \varphi(n)$.
   a. Show that the following functional is a distribution in $\mathcal{D}'(\mathbb{R}^1)$
   b. Find the weak derivative $\frac{d}{dx}F$

3. Let G be the distribution defined by
$$G(\varphi) = \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\varepsilon} \frac{f(x)}{x}\, dx + \int_{\varepsilon}^{+\infty} \frac{f(x)}{x}\, dx \right)$$
Prove
   a) that $w = \ln|x|$ is a locally integrable function in $\mathbb{R}^1$, and, hence it defines a regular distribution in $\mathcal{D}'(\mathbb{R}^1)$
   b) if W is the distribution in (a) then its weak derivative in $\mathcal{D}'(\mathbb{R}^1)$ is G.

4. Determine all solutions $u \in \mathcal{D}'(\mathbb{R}^1)$ of the equation $\frac{d}{dx}u = 0$. (Hint: consider $I(\varphi) = \int_{-\infty}^{+\infty} \varphi(x)dx$)

5. Define $u: \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ by $u(x) = \frac{1}{|x|}$. Show that $u \in L^1_{loc}(\mathbb{R}^3)$ and the associated distribution $\tilde{u}$ satisfies $\Delta u = C\delta_0$, where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ and δ_0 is the Dirac delta-function. Find C.

6. Let the function f is defined as
$$f(x_1, x_2) = \begin{cases} 1 & x_1, x_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$
Prove that f is a fundamental solution of the differential operator $Pu = \frac{\partial^2 u}{\partial x_1 \partial x_2}$.

7. Define the function $\psi: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ by
$$\psi = \begin{cases} \dfrac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} & t > 0 \\ 0 & t \le 0 \end{cases}$$
   a) Show that ψ is locally integrable in $\mathbb{R}^2$, and, thus, defines a distribution $\tilde{\psi} \in \mathcal{D}'(\mathbb{R}^2)$
   b) Prove that $\tilde{\psi}$ is a fundamental solution of the heat operator $\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ (you may use without proof: $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$).