Partiella differentialekvationer (D): Exercise problems (2008-09-04)

Vladimir Tkachev

1. Find an integral curve to the systems passing through the given point A:
   a. $\frac{dx}{yz} = \frac{dy}{z} = \frac{dz}{2x-y^2}, \qquad A = (1,1,1)$
   b. $\frac{dx}{y+z} = -\frac{dy}{z+x} = \frac{dz}{y-x}, \quad A = (1,1,2)$
2. Find a general solution for the following equations:
   a. $xu'_x + yu'_y + zu'_z = 0$
   b. $yu'_x + xu'_y = x - y$
   c. $yu'_x + xu'_y = x$
   d. $2xu'_x + (y - x)u'_y - x^2 = 0$
3. Find a solution to the homogeneous equation

$$u'_x + 2u'_y = 0$$

   which graph passes through the curve Γ with parameterization

$$x = s + s^2, \qquad y = 2s^2, \qquad z = s^2$$

4. Solve the given initial value problem and determine the values of x and y for which it exists:
   a. $xu'_x - yu'_y = 0, \quad u(x,1) = 2x$
   b. $xu'_x + u'_y = y\,, \quad u(x,0) = x^2$
   c. $u'_x - 2u'_y = u\,, \quad u(0,y) = y$
5. Solve the Cauchy problem

$$(1 + x^2)u'_x - 2xy\,u'_y = 0, \quad u(x, x + x^3) = h(x)$$

6. Find a solution to

$$xu'_x + yu'_y = 0, \qquad u(2,y) = y^2 + 1,$$

   which is defined for $x \geq 2$.

7. Solve the initial-value problem

$$xu'_x + yu'_y + u'_x u'_y = u, \quad u(x,0) = x^2$$

   by both characteristic method and the method of envelopes (find first affine solutions).
8. Find the solution to the equation

$$x{u'_x}^2 + yu'_y = 0, \quad u(x,1) = -x$$

9. Let $u(x,y)$, $y \geq 0$, be the weak solution to Burgers equation

$$uu'_x + u'_y = 0$$

   subject to the following initial condition:

$$u(x,0) = \begin{cases} 0, & x \leq 0 \\ -x, & 0 \leq x \leq 4 \\ -4, & x \geq 4 \end{cases}$$

   a. By using the Rankine-Hugoniot condition find the equation of the shock line
   b. Find an exact form of the weak solution and draw its characteristic

10. (McOwen 1.3.5)
11. Prove that the only solutions in all $\mathbb{R}^2$ to the equation

$$u^5 u'_x + u'_y = 0$$

   are the constants.
   (*Hint*: find the characteristic lines and discuss when they form a non-intersecting family)