Partiella differentialekvationer (D): Exercise problems (2008-09-22)

1. Determine all characteristic curves to the following equations and transform the equation to normal form in the given set:
   a. $u''_{xx} + 5u''_{xy} + 6u''_{yy} = 0$
   b. $x^2u''_{xx} - u''_{yy} = u$, for $x \neq 0$
   c. $y^2u''_{xx} - x^2u''_{yy} = 0$, for $x > 0, y > 0$

2. Reduce the equation to normal form and find its general solution
$$x^2u''_{xx} + 2xy\,u''_{xy} + y^2u''_{yy} = y.$$
3. Determine all characteristic curves to the equation
$$u''_{xx} - 2\sin x\,u''_{xy} - \cos^2 x\; u''_{yy} - \cos x\; u'_y = 0$$
and transform it to normal form. Find the general solution

4. Consider the following equations and answer the questions below:
   (i) $2yu''_{xx} - 2y^4u''_{yy} - 3y^3\,u'_y = 0,$
   (ii) $2u''_{xx} - 2y^3u''_{yy} - 3y^2\,u'_y = 0,$
   (iii) $\frac{3}{4}u''_{xx} - 2yu''_{xy} + y^2u''_{yy} + \frac{1}{2}\,u'_x = 0,$
   Study the following problems for each equation: (a) Where is the equation hyperbolic? (b) Determine the characteristic curves. (c) Transform the equation to canonical form where this is possible. (d) Determine its general solution in the domain where it is hyperbolic.

5. Solve the initial value problem $u''_{tt} - c^2u''_{xx} = 0, x > 0, y > 0$, subject to the initial condition:
   a. $u(0,t) = 1$ for $t > 0$; $u(x,0) = 1$, $u'_t(x,0) = \cos x - 1$ for $x > 0$.
   b. $u(0,t) = 0$ for $t > 0$; $u(x,0) = 0$, $u'_t(x,0) = 2cx\,e^{-x}$ for $x > 0$.
   c. $u(0,t) = 1$ for $t > 0$; $u(x,0) = e^{-x^2}$, $u'_t(x,0) = 0$ for $x > 0$.
   d. $u(0,t) = 0$ for $t > 0$; $u(x,0) = x^4$, $u'_t(x,0) = \sin x$ for $x > 0$.

6. Solve the differential equation $u''_{tt} = u''_{xx} + u'_x + u'_t$ with the initial conditions $u(x,0) = x$, $u'_t(x,0) = 0$. *Hint*: Apply the same method as for the wave equation.

7. Solve the differential equation $u''_{tt} = u''_{xx} - u'_t - k^2u$, $k \in \mathbb{R}$, satisfying the initial conditions $u(x,0) = h(x)$, $u'_t(x,0) = 0$, where $h(x) \in C^2(\mathbb{R})$.

8. Let $\Omega = \{(x,y):\ 0 \leq x \leq \pi,\ 0 \leq y \leq \pi\}$. Solve the boundary problem
$$u''_{xx} + u''_{yy} = 0, \qquad u(0,y) = u(\pi,y) = u(x,0) = 0, \qquad u(x,\pi) = 2\sin x - \sin 2x.$$

9. Let $\Omega = \{(x,y):\ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ and u is a solution of $\Delta u = 0$ in Ω. Solve the Dirichlet problem
$$u(x,0) = \sin 2\pi x, \qquad u(x,1) = \sin 3\pi x + x, \qquad u(1,y) = y, \qquad u(0,y) = 0.$$

10. Find the solution to the initial-value problem $u''_{tt} = u''_{xx} + u''_{yy} + u''_{zz}$, $u(x,y,z,0) = x^2 + y^2$, $u'_t(x,y,z,0) = 0$ using Kirchhoff's formula.
11. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfies
$$\Delta u - u^3 = -u$$
in Ω and vanishes on the boundary: $u|_{\partial\Omega} = 0$. By using the maximum principle show that $-1 \leq u \leq 1$.