Solutions to some problems from list 3, part II

Problem 4. *Determine all solutions* $u \in \mathcal{D}'(\mathbb{R}^1)$ *of the equation* $\frac{d}{dx}u = 0$*. (Hint: consider* $I(\varphi) = \int_{-\infty}^{+\infty} \varphi(x)dx$*).*

Solution. First notice that $I \in \mathcal{D}'(\mathbb{R}^1)$. Indeed, one can show this directly by definition, but a simpler way is to see this is to write

$$I(\varphi) = \int_{-\infty}^{+\infty} g(x)\varphi(x)dx,$$

that is I is a regular distribution generated by function $g \equiv 1$ which is obviously locally integrable in $\mathbb{R}^1$.

Now, we notice that for any test function φ we have

$$I(\varphi') = \int_{-\infty}^{+\infty} \varphi'(x)dx = 0$$

Hence, by definition of weak derivative

$$DI(\varphi) = -I(\varphi') = 0 \quad \Longrightarrow \quad DI = 0$$

Let us prove that any solution $u \in \mathcal{D}'(\mathbb{R}^1)$ of $Du = 0$ has the form $u(\varphi) = cI(\varphi)$ for some constant $c \in \mathbb{R}$. To this aim, let us denote by $\varphi_0 \in \mathcal{D}(\mathbb{R}^1)$ some test function for which $I(\varphi_0) \neq 0$ (for instance, let φ_0 be positive somewhere and non-negative everywhere; clearly such a function does exist). Then for any test function φ we can find $k \in \mathbb{R}$ such that

$$I(\varphi - k\varphi_0) = 0 \qquad (*)$$

Namely, $k = \frac{I(\varphi)}{I(\varphi_0)}$. It follows from the last identity that function $\varphi - k\varphi_0$ has an anti-derivative in $\mathcal{D}(\mathbb{R}^1)$. Indeed, set

$$\psi(x) = \int_{-\infty}^{x} \big(\varphi(t) - k\varphi_0(t)\big)dt.$$

Then $\psi(x) = 0$ for $x < -M$, where $[-M, M]$ is an interval containing both the support of φ and the support of φ_0. But $\psi(x) = 0$ also and for $x > M$ because we have (*) above. That is (since $\psi(x)$ is infinitely differentiable)

$$\psi(x) \in \mathcal{D}(\mathbb{R}^1).$$

But $\psi' = \varphi - k\varphi_0$, whence

$$0 = Du(\psi) = -u(\psi') = -u(\varphi) + ku(\varphi_0) = -u(\varphi) + u(\varphi_0) \cdot \frac{I(\varphi)}{I(\varphi_0)}.$$

Hence we obtained that $u(\varphi) = \frac{u(\varphi_0)}{I(\varphi_0)} I(\varphi)$ for any $\mathcal{D}(\mathbb{R}^1)$, therefore that $u = cI$ for $c = \frac{u(\varphi_0)}{I(\varphi_0)}$ ∎

Problem 5. *Define* $u: \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ *by* $u(x) = \frac{1}{|x|}$. *Show that* $u \in L^1_{loc}(\mathbb{R}^3)$ *and the associated distribution* $\tilde{u}$ *satisfies* $u = C\delta_0$ *, where* $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ *and* δ_0 *is the Dirac delta-function. Find* C*.*

Solution. First show that $u(x) = \frac{1}{|x|} \in L^1_{loc}(\mathbb{R}^3)$. It suffices to prove that $\frac{1}{|x|} \in L^1(B)$, where B is the unit ball. In its turn, the latter integrability is equivalent convergence of

$$\int_{B\setminus B_\epsilon} \frac{dx}{|x|}$$

Where $B_\epsilon = \{x \in \mathbb{R}^3 : |x| < \epsilon\}$. But the last integral converges:

$$\lim_{\epsilon\to 0} \int_{B\setminus B_\epsilon} \frac{dx}{|x|} \quad = \lim_{\epsilon\to 0} 4\pi \int_\epsilon^1 \frac{t^2 dt}{t} = \lim_{\epsilon\to 0} 2\pi(1-\epsilon^2) = 2\pi$$

Hence $\frac{1}{|x|} \in L^1_{loc}(\mathbb{R}^3)$ is proved.

In order to wind Δu we recall that on the level of distributions we have

$$\Delta\tilde{u}(\varphi) = (-1)^2\, \tilde{u}(\Delta\varphi), \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^3).$$

Denoting by B_R some ball, large enough to contain the support of a given test function $\varphi \in \mathcal{D}(\mathbb{R}^3)$. Notice also that $\Delta\frac{1}{|x|} = 0$ for $x \neq 0$ (recall that function $|x|^{n-2}$ is harmonic in $\mathbb{R}^n$). Then applying Green formula for annular domain $B_R \setminus B_\epsilon$ we obtain

$$(\Delta\tilde{u})(\varphi) = \tilde{u}(\Delta\varphi) \equiv \lim_{\epsilon\to 0} \int_{B_R\setminus B_\epsilon} \frac{\Delta\varphi}{|x|}\, dx = \lim_{\epsilon\to 0} \int_{B_R\setminus B_\epsilon} \left(\frac{1}{|x|}\Delta\varphi - \varphi\, \Delta\frac{1}{|x|}\right) dx$$

$$= \lim_{\epsilon\to 0} \int_{\partial B_R - \partial B_\epsilon} \left(\frac{1}{|x|}\varphi'_\nu - \varphi\, \left(\frac{1}{|x|}\right)'_\nu\right) dS = -\lim_{\epsilon\to 0} \int_{\partial B_\epsilon} \left(\frac{1}{|x|}\varphi'_\nu - \varphi\, \left(\frac{1}{|x|}\right)'_\nu\right) dS \quad (**)$$

Here we denoted by $(\cdot)'_\nu$ the normal derivative along the corresponding sphere, and dS stands for the area measure. We also applied the fact that $\varphi = 0$ on ∂B_R.

Furthermore we have (since $|x| = \epsilon$ on ∂B_ϵ)

$$\left|\int_{\partial B_\epsilon} \frac{1}{|x|}\varphi'_\nu\, dS\right| = \frac{1}{\epsilon}\left|\int_{\partial B_\epsilon} \varphi'_\nu dS\right| \le \frac{C}{\epsilon}\cdot 4\pi\epsilon^2 = 4\pi\epsilon = o(1), \qquad \epsilon \to 0.$$

Here $C = \max_{B_R} |\nabla\varphi|$ (we applied the inequality $|\varphi'_\nu| \le |\nabla\varphi|$). Thus

$$\lim_{\epsilon\to 0} \int_{\partial B_\epsilon} \frac{1}{|x|}\varphi'_\nu\, dS = 0.$$

The second integral in formula (**) above can be found explicitly. We notice that the normal derivative is

$$\left(\frac{1}{|x|}\right)'_\nu = \langle \nabla \frac{1}{|x|}, \nu \rangle = -\langle \frac{x}{|x|^3}, \nu \rangle = -\langle \frac{x}{|x|^3}, \frac{x}{|x|} \rangle = -\frac{1}{|x|^2},$$

Hence by continuity of φ at the origin we obtain

$$\lim_{\epsilon \to 0} \int_{\partial B_\epsilon} \varphi \left(\frac{1}{|x|}\right)'_\nu dS = -\lim_{\epsilon \to 0} \int_{\partial B_\epsilon} \frac{\varphi}{|x|^2} dS = -\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int_{\partial B_\epsilon} \varphi \, dS = -4\pi\varphi(0)$$

Thus we obtain

$$(\Delta \tilde{u})(\varphi) = -4\pi\varphi(0) = -4\pi\delta_0(\varphi)$$

which finally proves that $\Delta \tilde{u} = -4\pi\delta_0$.■

Problem 7. *Define the function* $\psi: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ *by*

$$\psi = \begin{cases} \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

a) *Show that* ψ *is locally integrable in* $\mathbb{R}^2$*, and, thus, defines a distribution* $\tilde{\psi} \in \mathcal{D}'(\mathbb{R}^2)$

b) *Prove that* $\tilde{\psi}$ *is a fundamental solution of the heat operator* $\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ *(you may use without proof:* $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$ *).*

Solution. We briefly describe the basic steps (the proof follows the same argument as solution of Problem 5 above).

In order to see that ψ is locally integrable in $\mathbb{R}^2$ it suffices to notice that

$$0 \leq \psi \leq \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \leq \frac{1}{2\sqrt{\pi t}}$$

It is easy to see that $\frac{1}{2\sqrt{\pi t}}$ is integrable in squares

$$Q = \{|x| \leq M, |t| \leq M\}, \qquad M > 0$$

Hence ψ is locally integrable in $\mathbb{R}^2$. Denote by $\tilde{\psi}$ the corresponding distribution.

Consider any test function $\phi \in \mathcal{D}(\mathbb{R}^2)$ and denote by $Q_\epsilon = \{|x| \leq M, t > \epsilon\}$ the square which contains the support of ϕ. Denote by L the heat operator $\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ and notice that

$$L\left(\frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}\right) = 0$$

for $t > 0$. Then, taking into account the definition of weak derivative we obtain

$$(L\tilde{\psi})(\phi) \equiv \left(\frac{\partial \tilde{\psi}}{\partial t} - \frac{\partial^2 \tilde{\psi}}{\partial x^2}\right)(\phi) = \tilde{\psi}\left(-\frac{\partial \phi}{\partial t} - \frac{\partial^2 \phi}{\partial x^2}\right) = -\lim_{\epsilon \to 0} \int_{Q_\epsilon} \left(\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2}\right) \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} dx$$

We find the last integral by splitting it into two parts:

$$I(\epsilon) = \int_{\epsilon}^{M} dt \int_{-M}^{M} \left(\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} \right) \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}\, dx =: I_1 + I_2$$

Denote $h(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$ we see that

$$I_1 = \int_{\epsilon}^{M} dt \int_{-M}^{M} h(x,t)\ \phi''_{xx} dx$$

and integrating the inner integral by parts two times (recall that $\phi(\pm M, t) = \phi(x, \pm M) = 0$, hence the boundary values are equal to zero) we ibtain:

$$\int_{-M}^{M} h\ d\phi'_x = \phi'_x h|_{x=-M}^{x=M} - \int_{-M}^{M} h'_x\ \phi'_x dx = \int_{-M}^{M} h''_{xx}\ \phi\, dx$$

Thus

$$I_1 = \int_{\epsilon}^{M} dt \int_{-M}^{M} h''_{xx} \phi\ dx$$

Similarly,

$$I_2 = \int_{\epsilon}^{M} dt \int_{-M}^{M} h\ \phi'_t dx = \int_{-M}^{M} dx \int_{\epsilon}^{M} h\, d\phi\ = \int_{-M}^{M} dx \left(h\phi|_{t=\epsilon}^{t=M} - \int_{\epsilon}^{M} h'_t \phi\, dt \right) =$$

$$= -\int_{-M}^{M} h(x,\epsilon)\phi(x,\epsilon) dx - \int_{-M}^{M} dx\ \int_{\epsilon}^{M} h'_t \phi\, dt$$

Summarizing

$$I(\epsilon) = I_1 + I_2 = -\int_{-M}^{M} h(x,\epsilon)\phi(x,\epsilon) dx - \int_{-M}^{M} dx\ \int_{\epsilon}^{M} (h'_t - h''_{xx})\phi\, dt\ =$$

$$= -\int_{-M}^{M} h(x,\epsilon)\phi(x,\epsilon) dx = -\int_{-\infty}^{\infty} h(x,\epsilon)\phi(x,\epsilon) dx$$

(recall that $(h'_t - h''_{xx} = 0)$. Setting $x = 2\sqrt{\epsilon} y$ we find

$$I(\epsilon) = -\int_{-\infty}^{\infty} \frac{\phi(x,\epsilon)}{2\sqrt{\pi\epsilon}} e^{-\frac{x^2}{4\epsilon}} dx = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} \phi\big(2\sqrt{\epsilon} y, \epsilon\big) dy$$

and since $\int_{\mathbb{R}} e^{-y^2} dy = \sqrt{\pi}$, we obtain

$$\lim_{\epsilon \to 0} I(\epsilon) = -\phi(0,0),$$

That is $L\tilde{\psi} = \delta_0$.■