Solutions to some problems from list 3, part I

Problem 1. *Prove the Leibniz rule for distributions: if $F \in \mathcal{D}'(U)$ and $f \in C^\infty(U)$, where $U \in \mathbb{R}^n$ then*

$$D(f \cdot F) = D(f) \cdot F + f \cdot D(F)$$

for $D = \partial_{x_i}$. (Hint: apply the both sides to a test function)

Solution. Fr any test function $\varphi \in \mathcal{D}(U)$

$$D(f \cdot F)(\varphi) = (\text{def. of } D) = -(f \cdot F)(D\varphi) =$$

$$(\text{def. of multiplication of a distribution by function}) = -F(fD\varphi) =$$

$$= -F(D(f\varphi) - \varphi\, f') = DF(f\varphi) + F(\varphi\, f') = (f \cdot DF)(\varphi) + (f' \cdot F)(\varphi) =$$

$$= (f \cdot DF + f' \cdot F)(\varphi),$$

hence the required identity is proved.

Problem 2. *Let $F(\varphi) = \sum_{k=1}^{\infty} \varphi\,(k)$.*

a. *Show that the following functional is a distribution in $\mathcal{D}'(\mathbb{R}^1)$*
b. *Find the weak derivative $\frac{d}{dx}F$*

Solution.

1) First prove that our functional is well-defined for any test function. For any $\varphi \in \mathcal{D}(\mathbb{R})$ we denote by $[-M, M]$ an interval which contains the support: supp $\varphi \subset [-M, M]$. Then
$$F(\varphi) = \sum_{k=1}^{\infty} \varphi\,(k) = \sum_{k=1}^{[M]} \varphi\,(k)$$
where $[M]$ is the integer part of M (the maximal integer which is less or equal to M). Hence, the sum is well-defined.
2) We prove linearity of F. Take, for instance, two test functions $\varphi, \psi \in \mathcal{D}(\mathbb{R})$ and denote by $[-M, M]$ an interval which contains supports of φ, ψ. Then $supp\ \varphi + \psi$ is contained also in this interval. Hence we have
$$F(\varphi) = \sum_{k=1}^{[M]} \varphi\,(k), \qquad F(\psi) = \sum_{k=1}^{[M]} \psi\,(k), \qquad F(\varphi + \psi) = \sum_{k=1}^{[M]} (\varphi + \psi)\,(k)$$
On the other hand, the latter sum is equal to
$$F(\varphi + \psi) = \sum_{k=1}^{[M]} (\varphi + \psi)\,(k) = \sum_{k=1}^{[M]} \varphi\,(k) + \sum_{k=1}^{[M]} \psi\,(k) = F(\varphi) + F(\psi),$$
and it follows that F is an additive functional. Similarly one shows that $F(\lambda\varphi) = \lambda F(\varphi)$.

3) We show that our function is continuous in $\mathcal{D}'(\mathbb{R}^1)$, that is *for any function* $\varphi \in \mathcal{D}(\mathbb{R})$ *and any sequence* $(\varphi_n)_{n\geq 1} \subset \mathcal{D}(\mathbb{R})$ *such that* $\varphi_n \xrightarrow{\mathcal{D}} \varphi$ *we have* $F(\varphi_n) \to F(\varphi)$.
   Indeed, we find (by definition of the convergence $\varphi_n \xrightarrow{\mathcal{D}} \varphi$) a compact set, say, interval $[-M, M]$ which contains all supports of φ_n, $n \geq 1$. Then, as above in (2) we can write
$$F(\varphi_n) = \sum_{k=1}^{[M]} \varphi_n\,(k).$$
   On the other hand, we know also that for any index $p \geq 0$ the derivatives converges uniformly:
$$\left\|\varphi_n^{(p)} - \varphi^{(p)}\right\|_\infty \to 0, \qquad \text{as } n \to \infty.$$
   It suffices to consider only $p = 0$: $\|\varphi_n - \varphi\|_\infty \to 0$. We have
$$|F(\varphi_n) - F(\varphi)| = \left|\sum_{k=1}^{[M]} \varphi_n\,(k) - \sum_{k=1}^{[M]} \varphi\,(k)\right| \leq \sum_{k=1}^{[M]} |\varphi_n(k) - \varphi(k)| \leq [M] \cdot \|\varphi_n - \varphi\|_\infty$$
   But the latter expression is converges to zero. Hence we have proved that $F \in \mathcal{D}'(\mathbb{R}^1)$.
4) Now we find weak derivative of F. We have by definition
$$DF(\varphi) = -F(\varphi') = \sum_{k=1}^{[M]} \varphi'\,(k) = \sum_{k=1}^{\infty} \varphi'\,(k)$$
   Hence $DF(\varphi) = \sum_{k=1}^{\infty} \varphi'\,(k)$.

Problem 3. *Let G be the distribution defined by*
$$G(\varphi) = \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon} \frac{\varphi}{x}\,dx + \int_{\varepsilon}^{+\infty} \frac{\varphi}{x}\,dx\right)$$

Prove

a) *that $w = \ln|x|$ is a locally integrable function in $\mathbb{R}^1$, and, hence it defines a regular distribution in $\mathcal{D}'(\mathbb{R}^1)$*
b) *if W is the distribution in (a) then its weak derivative in $\mathcal{D}'(\mathbb{R}^1)$ is G.*

Solution. (a) is a standard exercise in elementary Lebesgue theory. For any compact subset of $\mathbb{R}$ which does not contain 0 our function is continuous, therefore, is integrable there. If a compact set contains 0, one uses standard cut-off argument and the fact that the integral
$$\int_0^M \ln x\; dx$$
converges for any finite M.

(b) To find the weak derivative we consider an arbitrary $\varphi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}\varphi \subset [-M, M]$ and notice that
$$DW(\varphi) = -W(\varphi') = -\int_{-\infty}^{+\infty} \varphi'(x)\ln|x| = -\int_{-M}^{+M} \varphi'(x)\ln|x| \;=$$

$$= -\lim_{\varepsilon\to 0}\left(\int_{-M}^{-\varepsilon}\varphi'(x)\ln(-x)\,dx + \int_{\varepsilon}^{M}\varphi'(x)\ln x\;dx\right) =$$

$$= -\lim_{\varepsilon\to 0}\left(\int_{-M}^{-\varepsilon}\ln(-x)\;d\varphi(x) + \int_{\varepsilon}^{M}\ln x\;d\varphi(x)\right)$$

(by integrating by parts and by using $\varphi(-M) = \varphi(M) = 0$)

$$= -\lim_{\varepsilon\to 0}\left(\varphi(-\varepsilon)\ln\varepsilon - \int_{-M}^{-\varepsilon}\varphi(x)\;\frac{dx}{x} - \varphi(\varepsilon)\ln\varepsilon - \int_{\varepsilon}^{M}\varphi(x)\ln x\;dx\right)$$

$$= \lim_{\varepsilon\to 0}\left(\big(\varphi(\varepsilon) - \varphi(-\varepsilon)\big)\ln\varepsilon + \int_{-M}^{-\varepsilon}\varphi(x)\;\frac{dx}{x} + \int_{\varepsilon}^{M}\varphi(x)\;\frac{dx}{x}\right)$$

$$= \lim_{\varepsilon\to 0}\left(\int_{-M}^{-\varepsilon}\varphi(x)\;\frac{dx}{x} + \int_{\varepsilon}^{M}\varphi(x)\ln x\;dx\right)$$

Here we used the differentiability of φ at zero:

$$\varphi(\varepsilon) - \varphi(-\varepsilon) \sim C\cdot\varepsilon \quad\Rightarrow\quad \big(\varphi(\varepsilon) - \varphi(-\varepsilon)\big)\ln\varepsilon = o(1)$$

Thus, we obtained the required formula.

Problem 6. *Let the function f is defined as*

$$f(x_1,x_2) = \begin{cases} 1 & x_1,x_2 > 0 \\ 0 & otherwise \end{cases}$$

Prove that f is a fundamental solution of the differential operator $Pu = \frac{\partial^2 u}{\partial x_1 \partial x_2}$.

Solution. Our distribution is regular and it is defined for any $\varphi \in \mathcal{D}(\mathbb{R}^2)$ by

$$F(\varphi) = \int_{\mathbb{R}^2} f(x_1,x_2)\varphi(x_1,x_2)\,dx_1dx_2 = \int_{\mathbb{R}_+^2}\varphi(x_1,x_2)\,dx_1dx_2$$

where $\mathbb{R}_+^2 = \{(x_1,x_2)\colon\ x_1,x_2 > 0\}$. We find $M > 0$ such that

$$\operatorname{supp}\varphi \subset [-M,M]\times\ [-M,M].$$

Let $L = \frac{\partial^2}{\partial x_1\partial x_2}$. Then by definition (notice that L has an even order) and taking into account that $\varphi(x_1,\pm M) = \varphi(\pm M,x_2) = 0$, we obtain

$$LF\;(\varphi) = \int_{\mathbb{R}_+^2} L\varphi\;dx_1dx_2 = \int_0^M dx_1\int_0^M \frac{\partial^2\varphi}{\partial x_1\partial x_2}\;dx_2 = -\int_0^M \varphi'_{x_1}(x_1,0)dx_1 = \varphi(0,0).$$

Hence h is the fundamental solution of the operator $L = \frac{\partial^2}{\partial x_1\partial x_2}$ and our constant c is equal to 1: $LF = \delta_0$.