Some solutions and hints for the problem set 1, part II

Problem 6. *Find a solution to* $xu_x' + yu_y' = 0$, $u(2, y) = y^2 + 1$, *which is defined for* $x \geq 2$.

Solution. The characteristics are found as $x = C_1 e^t,\ y = C_2 e^t,\ u = C_3$. The parameterization of our initial conditions is

$$\Gamma: \quad x(s) = 2, \qquad y(s) = s, \qquad u(s) = s^2 + 1.$$

We verify the non-characteristic condition $x_t' y_s' - x_s' y_t' \neq 0$:

$$x_t' y_s' - x_s' y_t' = 2 \cdot 1 - 0 \cdot s = 2 \neq 0$$

After the substitution of the initial condition we obtain $C_1 = 2,\ C_2 = s$ and $C_3 \equiv s^2 + 1$. We have the parametric representation of the found solution:

$$x = 2e^t, \qquad y = se^t = \frac{sx}{2}, \qquad u = s^2 + 1.$$

We find $u = \left(\frac{2y}{x}\right)^2 + 1$. Obviously, the solution is well-defined for $x \geq 2$.

Problem 7. *Solve the initial-value problem* $xu_x' + yu_y' + u_x' u_y' = u,\ u(x, 0) = x^2$ *by both characteristic method and the method of envelopes (find first affine solutions).*

Solution. Unfortunately, a misprint has crept into the printed version of the problem list: instead of the initial condition $u(x, \mathbf{0}) = x^2$ there was $u(x, \mathbf{1}) = x^2$. This small thing makes solution by using of the envelope method harder. However, I give full solution of the "misprinted" problem in **Appendix** below. Now I solve the corrected problem.

The envelope method*.* We try first the affine solution $v = a + bx + cy + dxy$. After substitution

$$x(b + dy) + y(c + dx) + (b + dy)(c + dx) = a + bx + cy + dxy$$

and equating the coefficients of $1, x, y, xy$, we obtain

$$a = bc, \qquad b(1 + d) = b, \qquad c(1 + d) = c, \qquad d(d + 1) = 0,$$

which yields a non-trivial solution $a = bc$ and $d = 0$. Hence the trial affine solution is $v = bc + bx + cy$. Now we take $c = kb$, where k will be chosen later:

$$v(x, y; b) = kb^2 + bx + kby$$

Then the envelope equation is $\frac{\partial}{\partial b} v(x, y; b) = 0$, that is $2bk + x + ky = 0$, hence $b = -\frac{x+ky}{2k}$. Substitution of the found parameter into $v(x, y; b)$ gives

$$v(x, y; b) = k\left(-\frac{x + yk}{2k}\right)^2 - \frac{x + yk}{2k}(x + ky) = \frac{1}{4k} \cdot (x + ky)^2$$

Finally, verify the initial condition $u(x, 0) = x^2$:

$$u(x, 0) = \frac{1}{4k} \cdot x^2 = x^2 \qquad \Rightarrow \qquad 4k = 1$$

Hence $k = \frac{1}{4}$ and it follows that $u = \frac{1}{4k} \cdot (x + ky)^2 = \left(x + \frac{y}{4}\right)^2$.

The method of strips. In our notation: $F = xp + yq + pq - z$, where $p = u_x^{'}, q = u_y^{'}, z = u$. Hence the extended system of characteristic equations is

$$\dot{x} = F_p^{'} = x + q, \qquad \dot{y} = F_q^{'} = y + p, \qquad \dot{z} = pF_p^{'} + qF_q^{'} = xp + yq + 2pq$$

and

$$\dot{p} = -F_x^{'} - F_z^{'} p = -p + p = 0, \qquad \dot{q} = -F_y^{'} - F_z^{'} q = -q + q = 0.$$

We have $p = C_1$ and $q = C_2$. Our Cauchy condition is given by

$$\Gamma: \quad x = x_0(s) = s, \qquad y = y_0(s) = 0, \qquad z = z_0(s) = s^2$$

hence, applying the strip condition $\frac{d}{ds} z_0(s) = p_0(s) \cdot \frac{dx_0}{ds} + q_0(s) \cdot \frac{dy_0}{ds}$ we get an additional equation:

$$2s = C_1 \cdot 1 + C_2 \cdot 0 = C_1 \qquad \Rightarrow \qquad C_1 = 2s$$

And applying our initial condition for $F = 0$ we get

$$0 = x(0)p_0 + y(0)q_0 + p_0 q_0 - z(0) = sC_1 + C_1 C_2 - s^2 = s^2 + 2sC_2$$

whence $C_2 = -\frac{s}{2}$. Moreover, from the ODE for $\dot{z}$ we find that

$$\dot{z} = xp + yq + 2pq = (\text{since } F = 0) = z + pq = z + C_1 C_2$$

which gives $z = -C_1 C_2 + C_3 e^t$. Substituting the found constants into the equation we obtain

$$z = s^2 + C_3 e^t = s^2 + C_3 e^t$$

and comparing this with $z(0) = s^2$ we find $C_3 = 0$, that is $z = s^2$.

Similarly we solve two remaining equations for x and y. For example, for x we have

$$\dot{x} = x + q = x + C_2 = x - \frac{s}{2} \qquad \Rightarrow \qquad x = C_4 e^t + \frac{s}{2}$$

where C_4 is found from the initial conditions as $C_4 = s - \frac{s}{2} = \frac{s}{2}$. Thus our solution takes the form

$$x = \frac{s}{2}(1 + e^t), \qquad y = 2s(e^t - 1), \qquad z = s^2$$

It follows from the first two equations that $s = x - \frac{y}{4}$, hence $z = \left(x - \frac{y}{4}\right)^2$ is the required solution.

Problem 8. *Find the solution to the equation* $xu_x'^2 + yu_y' = 0, \quad u(x, 1) = -x$.

Solution. We solve the equation by method of strips. We have $F = xp^2 + yq$ and the initial conditions

$$\Gamma: \quad x = x_0(s) = s, \qquad y = y_0(s) = 1, \qquad z = z_0(s) = -s$$

Then

$$\dot{x} = F_p^{'} = 2px, \qquad \dot{y} = F_q^{'} = y, \qquad \dot{z} = pF_p^{'} + qF_q^{'} = 2xp^2 + yq = -yq$$

The second equations gives $y = C_1 e^t$, and after initial conditions we find $y = e^t$. The extended system

$$\dot{p} = -F_x' - F_z' p = -p^2, \qquad \dot{q} = -F_y' - F_z' q = -q$$

gives $p = \frac{1}{t+C_2}$ and $q = C_3 e^{-t}$. We find also from $\frac{d}{ds} z_0(s) = p_0(s) \cdot \frac{dx_0}{ds} + q_0(s) \cdot \frac{dy_0}{ds}$ that

$$-1 = \frac{1}{C_2} \cdot 1 + C_3 \cdot 0 \qquad \Rightarrow \quad C_2 = -1$$

and rewriting initial condition for $F = 0$ we get

$$0 = x(0) p_0^2 + y(0) q_0 = s \left(\frac{1}{C_2}\right)^2 + 1 \cdot C_3 \qquad \Rightarrow \qquad C_3 = -s$$

Hence $p = \frac{1}{t-1}$ and $q = -se^{-t}$. We have $\dot{x} = 2px = \frac{2x}{t-1}$, whence $x = C_4(t-1)^2$. The IC[1] gives

$$x = s(t-1)^2$$

Similarly, $\dot{z} = -yq = se^{-t}\, y = se^{-t} \cdot e^t = s$. Hence $z = st + C_5$, and applying IC we find $z = st - s$.

Summarizing, we have

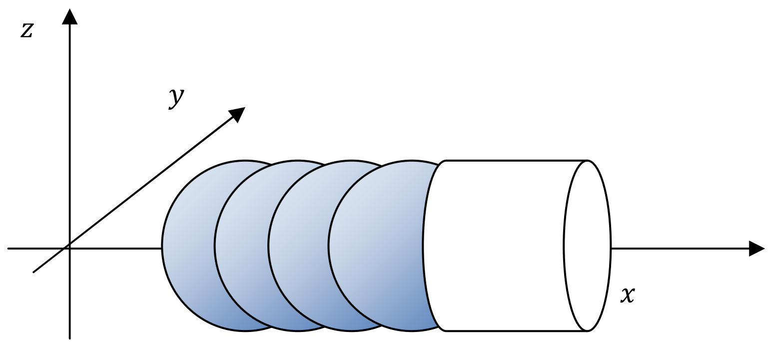
$$x = s(t-1)^2, \qquad y = e^t, \qquad z = st - s$$

which implies finally

$$z = \frac{x}{\ln y - 1}$$

Problem 10. Show that the family of spheres S_a given by $(x-a)^2 + y^2 + z^2 = 1$ has as its envelope the unit cylinder $y^2 + z^2 = 1$.

Solution. Geometrically this statement is transparent: one moves the sphere S_0: $x^2 + y^2 + z^2 = 1$ along the x-axis, hence it envelope is the surface of the cylinder $y^2 + z^2 = 1$:



Analytically, however, the statement is equally easy to prove: we set the envelope equation $F_a' = 0$ for the defining function $F(x, y, z; a) \coloneqq (x-a)^2 + y^2 + z^2 - 1$. In other words, we obtain $x - a = 0$, hence $a = x$. Substituting this into the defining equation we obtain

$$F(x = a, y, z; a) \coloneqq (a-a)^2 + y^2 + z^2 - 1 = y^2 + z^2 - 1 = 0$$

which is the required equation.

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[1] IC = initial condition

Problem 11. *Prove that the only solutions in all* $\mathbb{R}^2$ *to the equation* $u^5 u_x' + u_y' = 0$ *are the constants.*

Solution. Let $u(x,y)$ be any solution to this equation in the whole $\mathbb{R}^2$. The characteristics are given by

$$\dot{x} = u^5, \qquad \dot{y} = 1, \qquad \dot{u} = 0$$

That is

$$u = C, \qquad x = C^5 t + x_0, \quad y = t + y_0$$

In particular, the solution is constant along the lines parameterized by $x = C^5 t + x_0$ and $y = t + y_0$. We find the explicit equation of the lines:

$$x = C^5 (y - y_0) + x_0$$

Notice that, however, if there are at least two different values C_1 and C_2 which takes $u(x,y)$, then the above lines must intersect at some point in $\mathbb{R}^2$ (because the lines have different "slope-coefficients"). But in this case $u(x,y)$ takes values C_1 and C_2 at the intersection point, which contradicts single-valued character of $u(x,y)$. The contradiction follows, hence the solution must be a constant.

Appendix*: solution of problem 6 by the method of strips.*

We have in our notation: $F = xp + yq + pq - z$, where $p = u_x'$, $q = u_y'$, $z = u$. Hence the extended system of characteristic equations is

$$\dot{x} = F_p' = x + q, \qquad \dot{y} = F_q' = y + p, \qquad \dot{z} = pF_p' + qF_q' = xp + yq + 2pq$$

and

$$\dot{p} = -F_x' - F_z' p = -p + p = 0, \qquad \dot{q} = -F_y' - F_z' q = -q + q = 0.$$

We have $p = C_1$ and $q = C_2$. Moreover, from the ODE for $\dot{z}$ we find

$$\dot{z} = xp + yq + 2pq = (\text{since } F = 0) = z + pq = z + C_1 C_2$$

which can be easily solved as $z = -C_1 C_2 + C_3 e^t$.

Our Cauchy condition is given by

$$\Gamma: \quad x = x_0(s) = s, \qquad y = y_0(s) = 1, \qquad z = z_0(s) = s^2$$

hence, applying the strip condition $\frac{d}{ds} z_0(s) = p_0(s) \cdot \frac{dx_0}{ds} + q_0(s) \cdot \frac{dy_0}{ds}$ we get an additional equation:

$$2s = C_1 \cdot 1 + C_2 \cdot 0 = C_1 \qquad \Rightarrow \qquad C_1 = 2s$$

And applying our initial condition for $F = 0$ we get

$$0 = x(0)p_0 + y(0)q_0 + p_0 q_0 - z(0) = sC_1 + C_2 + C_1 C_2 - s^2 = s^2 + C_2(1 + 2s)$$

whence $C_2 = -\frac{s^2}{1+2s}$. Substituting this into the equation for z we find that

$$z = -C_1 C_2 + C_3 e^t = -\frac{2s^3}{1 + 2s} + C_3 e^t$$

and comparing this with $z(0) = s^2$ we find

$$C_3 = s^2 + \frac{2s^3}{1+2s} = \frac{s^2(1+4s)}{1+2s}$$

Similarly we solve two remaining equations for x and y. For example, for x we have

$$\dot{x} = x + q = x + C_2 = x - \frac{s^2}{1+2s} \qquad \Rightarrow \qquad x = C_4 e^t + \frac{s^2}{1+2s}$$

where C_4 is found as $C_4 = s - \frac{s^2}{1+2s} = \frac{s(1+s)}{1+2s}$.

Thus our solution has the form

$$x = \frac{s(1+s)}{1+2s} e^t + \frac{s^2}{1+2s}, \qquad y = (2s+1)e^t - 2s, \qquad z = -\frac{2s^3}{1+2s} + \frac{s^2(1+4s)}{1+2s} e^t$$

Some standard elimination algebra yields then the following implicit representation of our solution:

$$-9y^3z + 9x^2y^2 - 3y^2z^2 + 30xy^2z + 32x^2zy + 8z^2xy - 54yz^2 + 24x^3y -$$

$$-54zxy + 16z^2x^2 + 16x^4 - 16z^3 - 72zx^2 + 24xz^2 - 32x^3z + 81z^2 = 0$$