Some solutions and hints for the problem set 1

Problem 1. *Find an integral curve to the systems passing through the given point A*:

a. $\frac{dx}{yz} = \frac{dy}{z} = \frac{dz}{2x-y^2}, \qquad A = (1,1,1)$

b. $\frac{dx}{y+z} = -\frac{dy}{z+x} = \frac{dz}{y-x}, \quad A = (1,1,2)$

Solution of 1b. We use the Lagrange method. Notice that the system of integral curves has the form

$$\dot{x} = y + z, \qquad \dot{y} = -z - x, \qquad \dot{z} = y - x.$$

Then comparing the sum of two first equations with the last equation yields

$$\dot{x} + \dot{y} = y - x = \dot{z}\,,$$

hence $x + y - z = C_1$. By using our initial condition (that is the fact that the curve is passing through $A = (1,1,2)$) we find $C_1 = 1 + 1 - 2 = 0$. Hence our curve lies in the plane

$$z = x + y.$$

Substituting this into the first two differential equations gives the following linear system in *two* unknown functions:

$$\dot{x} = x + 2y, \qquad \dot{y} = -2x - y,$$

which can be solved by the standard methods. But we can solve it as follows: multiplying by the right han sides we get:

$$0 = \dot{x}(2x + y) + \dot{y}\,(x + 2y) = \frac{d}{dt}(x^2 + y^2 + xy)$$

Hence $x^2 + y^2 + xy = C_2$ and it follows form the initial data that $C_2 = 1 + 1 + 1 = 3$. Finally we get the system of two equations

$$x^2 + y^2 + xy = 3, \qquad z = x + y$$

defining the required curve (geometrically, this is an ellipse in the plane $z = x + y$).

Problem 2. *Find a general solution for the following equations:*

c. $xu'_x + yu'_y + zu'_z = 0$
d. $yu'_x + xu'_y = x - y$
e. $yu'_x + xu'_y = x$
f. $2xu'_x + (y - x)u'_y - x^2 = 0$

Solution 2a. The characteristic equations are $\dot{x} = x,\ \dot{y} = y,\ \dot{z} = z,\ \dot{u} = 0$. Hence $u = const$ along the characteristics. Solving the first three equations we obtain

$$x = C_1 e^t, \qquad y = C_2 e^t, \qquad z = C_3 e^t,$$

whence $\frac{x}{y}$ and $\frac{x}{z}$ are constants along the characteristics. It follows that the general solution is a function of these two (functionally independent) expressions, i.e. the general solution has the form $u = F\left(\frac{x}{y}, \frac{x}{z}\right)$ for an arbitrary function $F(\xi, \eta)$.

Solution 2d. The characteristic equations are $\dot{x} = 2x,\ \dot{y} = y - x,\ \dot{u} = -x^2$. The first equation gives $x = C_1 e^{2t}$, hence we find from the third equation that

$$u = -\frac{C_1^2}{4} e^{4t} + C_2 = -\frac{x^2}{4} + C_2.$$

Substituting the found $x(t)$ into the second equation, $\dot{y} = y - C_1 e^{2t}$, and integrating it we obtain

$$y = C_3 e^t - C_1\, e^{2t} = C_3 e^t - x.$$

Eliminating the variable t we find $(y + x) = C_3 e^t$. Hence the characteristics are

$$4\,u + x^2 = \tilde{C}_2, \qquad x + y = \tilde{C}_3 \sqrt{x}$$

Hence the general solution is given implicitly by $F\left(4u + x^2, \frac{x+y}{\sqrt{x}}\right) = 0$ with an arbitrary function $F(\xi, \eta)$. In particular, one can obtain also explicit solution like $u = -\frac{x^2}{4} + f\left(\frac{x+y}{\sqrt{x}}\right)$.

Problem 3. *Find a solution to the homogeneous equation $u_x' + 2u_y' = 0$ which graph passes through the curve Γ with parameterization $x = s + s^2,\ y = 2s^2,\ z = s^2$.*

Solution. The characteristic equations are $\dot{x} = 1,\ \dot{y} = 2,\ \dot{u} = 0$, hence

$$x = C_1 + t, \qquad y = C_2 + 2t, \qquad u = C_3\ .$$

We have $x(0) = C_1 = s + s^2, y(0) = C_2 = 2s^2$ and $u|_\Gamma = C_3 = s^2$. Thus we obtain the following parameterization of our solution:

$$x = t + s + s^2, \qquad y = 2t + 2s^2, \qquad u = s^2.$$

Eliminating the new variables we obtain

$$y = 2t + 2s^2 = 2t + 2u \qquad \Rightarrow \qquad t = \frac{y}{2} - u$$

and $x = t + s + s^2 = \left(\frac{y}{2} - u\right) + \sqrt{u} + u$. Hence the required solution is

$$u = \left(x - \frac{y}{2}\right)^2.$$

Problem 4a. *Solve the given initial value problem and determine the values of x and y for which it exists:*

$$xu_x' - yu_y' = 0, \quad u(x,1) = 2x.$$

Solution. The characteristics equations are $\dot{x} = x,\ \dot{y} = -y,\ \dot{u} = 0$, so we have

$$x = C_1 e^t, \qquad y = C_2 e^{-t}, \qquad u = C_3.$$

The initial conditions may be parameterized as follows: $x = s, y = 1, u = 2s$. Hence we find the constants:

$$x(0) = C_1 = s, \qquad y(0) = C_2 = 1, \qquad u = C_3 = 2s.$$

We have $x = se^t,\ y = e^{-t},\ u = 2s$, which yields $xy = s = \frac{u}{2}$. Thus the solution is $u = 2xy$

Problem 4b. *Solve the given initial value problem and determine the values of x and y for which it exists:*

$$xu'_x + u'_y = y\,, \qquad u(x,0) = x^2$$

Solution. The characteristics are: $x = C_1e^t,\ y = t + C_2,\ u = \frac{t^2}{2} + C_2t + C_3$. The constants are found from the initial conditions as $C_1 = s, C_2 = 0,\ C_3 = s^2$. Hence the parametric form of our solution is

$$x = se^t, \qquad y = t, \qquad u = \frac{t^2}{2} + s^2$$

We have $s = xe^{-y}$, therefore $u = \frac{t^2}{2} + s^2 = \frac{y^2}{2} + x^2e^{-2y}$.

Problem 5. *Solve the Cauchy problem*

$$(1 + x^2)u'_x - 2xy\, u'_y = 0, \quad u(x, x + x^3) = h(x)$$

Solution. The characteristics are: $\dot{x} = 1 + x^2,\ \dot{y} = 2xy,\ u = const$. The initial condition is given by the parameterization

$$\Gamma: \qquad x = s, \qquad y = s{+}s^3, \qquad u = h(x),$$

The non-characteristic condition is

$$(s)' \cdot (2xy)|_\Gamma - (s + s^3)' \cdot (1 + x^2)|_\Gamma -$$

$$= 2s(s + s^3) - (1 + 3s^2)(1 + s^2)$$

$$= -s^4 - 2s^2 - 1 = -(1 + s^2)^2 \neq 0$$

Thus our initial condition is non-characteristic and the problem can be solved for any function $h(x)$. Dividing $\dot{y} = 2xy$ by $\dot{x} = 1 + x^2$ we find the ODE

$$\frac{dy}{dx} = \frac{2xy}{1 + x^2}$$

which can be solved by separation of variables:

$$\frac{dy}{y} = \frac{2x\,dx}{1 + x^2} \qquad \Rightarrow \qquad \ln y = \ln(1 + x^2) + C_1$$

Hence, $y = C_2(1 + x^2)$. Substituting the parametric initial conditions $x = s,\ y = s{+}s^3$ yields $C_2 = s$, and it follows that $y = s(1 + x^2)$.

From $u = const$ along the characteristics we see that $u = h(s)$. Therefore the required solution is

$$u = h(s) = h\left(\frac{y}{1 + x^2}\right).$$