Solutions to some problems from list 2

Problem 1a. *Determine all characteristic curves to the following equations and transform the equation to normal form in the given set:*

$$x^2u''_{xx} - u''_{yy} = u, x \neq 0$$

Solution. The equation for characteristic curves in non-parametric form is

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\pm 2x}{2x^2} = \pm\frac{1}{x}$$

Hence the equation is hyperbolic for $x \neq 0$ (the discriminant $b^2 - 4ac = 4x^2$ is positive there) and we find two solutions

$$y = \ln x + C_1, \qquad y = -\ln x + C_2\,.$$

We introduce the new coordinates

$$\lambda = C_2 = y + \ln x\,, \qquad \mu = C_1 = y - \ln x$$

and express the old derivatives:

$$u'_x = u'_\lambda \lambda'_x + u'_\mu \mu'_x = \frac{u'_\lambda - u'_\mu}{x}$$

$$u''_{xx} = -\left(\frac{1}{x^2}\right)(u'_\lambda - u'_\mu) + \left(\frac{1}{x^2}\right)(u''_{\lambda\lambda} - 2u'_{\mu\lambda} + u''_{\mu\mu})$$

And similarly

$$u'_y = u'_\lambda + u'_\mu$$

$$u''_{yy} = u''_{\lambda\lambda} + 2u'_{\mu\lambda} + u''_{\mu\mu}$$

Then substitution these relations into our equation yields

$$\frac{x^2}{x^2}(-u'_\lambda + u'_\mu + u''_{\lambda\lambda} - 2u'_{\mu\lambda} + u''_{\mu\mu}) - (u''_{\lambda\lambda} + 2u'_{\mu\lambda} + u''_{\mu\mu}) = u,$$

Combining terms we find the normal form:

$$4u'_{\mu\lambda} + u'_\lambda - u'_\mu + u = 0$$

Problem 2. *Reduce the equation to normal form and find its general solution*

$$x^2u''_{xx} + 2xy\,u''_{xy} + y^2u''_{yy} = y.$$

Solution. We follow the previous solution: $a = x^2, b = 2xy, c = y^2$ so that the discriminant is $b^2 - 4ac = 0$, hence our equation has parabolic type. The characteristic equation is

$$\frac{dy}{dx} = \frac{b}{2a} = \frac{2xy}{2x^2} = \frac{y}{x},$$

which implies $\frac{dy}{y} = \frac{dx}{x}$ and therefore $y = Cx$. We take

$$\lambda = \frac{y}{x}, \qquad \mu = x$$

(in the parabolic case one can choose the second variable arbitrarily). Hence

$$u'_x = u'_\lambda \lambda'_x + u'_\mu \mu'_x = -\frac{y}{x^2} u'_\lambda + u'_\mu$$

$$u''_{xx} = \frac{2y}{x^3} u'_\lambda - \frac{y}{x^2}\left(-\frac{y}{x^2} u''_{\lambda\lambda} + u''_{\mu\lambda}\right) + \left(-\frac{y}{x^2} u''_{\mu\lambda} + u''_{\mu\mu}\right) = \frac{2y}{x^3} u'_\lambda + \frac{y^2}{x^4} u''_{\lambda\lambda} - \frac{2y}{x^2} u''_{\mu\lambda} + u''_{\mu\mu}$$

$$u''_{xy} = -\frac{1}{x^2} u'_\lambda - \frac{y}{x^2}\left(\frac{1}{x} u''_{\lambda\lambda}\right) + \frac{1}{x} u''_{\lambda\mu}$$

$$u''_{yy} = \frac{1}{x^2} u''_{\lambda\lambda}$$

Substitution into $L[u] \coloneqq x^2 u''_{xx} + 2xy\, u''_{xy} + y^2 u''_{yy} - y = 0$ yields

$$L[u] = x^2\left(\frac{2y}{x^3} u'_\lambda + \frac{y^2}{x^4} u''_{\lambda\lambda} - \frac{2y}{x^2} u''_{\mu\lambda} + u''_{\mu\mu}\right) + 2xy\left(-\frac{1}{x^2} u'_\lambda - \frac{y}{x^2}\left(\frac{1}{x} u''_{\lambda\lambda}\right) + \frac{1}{x} u''_{\lambda\mu}\right) + \frac{y^2}{x^2} u''_{\lambda\lambda} - y$$
$$= x^2 u''_{\mu\mu} - y = 0$$

In the new coordinates, the latter equation reads as

$$u''_{\mu\mu} = \frac{y}{x^2} = \frac{\lambda}{\mu}$$

The integration yields

$$u'_\mu = \lambda \ln \mu + C(\lambda)$$

$$u = \lambda(\mu \ln \mu - \mu) + C_1(\lambda)$$

Hence the general solution has the form

$$u = \frac{y}{x}(x \ln x - x) + f\left(\frac{y}{x}\right) = y \ln x - y + f\left(\frac{y}{x}\right),$$

where $f(t)$ is an arbitrary function of t.

Problem 4 (i). *Consider the following equations and answer the questions below:*

$$2yu''_{xx} - 2y^4 u''_{yy} - 3y^3\, u'_y = 0,$$

Study the following problems for each equation: (a) Where is the equation hyperbolic? (b) Determine the characteristic curves. (c) Transform the equation to canonical form where this is possible. (d) Determine its general solution in the domain where it is hyperbolic.

Solution. (a) In our case $a = 2y, b = 0, c = -2y^4$. Hence $D = b^2 - 4ac = 16y^5$ and the equation is *hyperbolic* when $y > 0$ (and *elliptic* when $y < 0$).

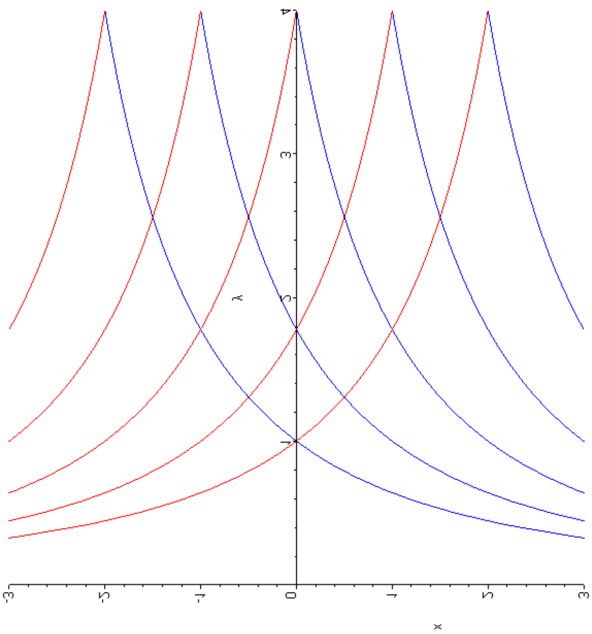
(b) To find the characteristic curves we write the corresponding ODE in the hyperbolic domain $y > 0$:

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{0 \pm \sqrt{16y^5}}{4y} = \pm y^{\frac{3}{2}}.$$

The latter yields

$$\frac{dy}{y^{3/2}} = \pm dx \;\Rightarrow\; -\frac{2}{\sqrt{y}} = \pm x + C$$

The characteristic curves are hyperbolas $-\frac{2}{\sqrt{y}} = x + C$ and $\frac{2}{\sqrt{y}} = -x + C$ (in the upper half-plane):



(c) In these new coordinates ⇒ $\lambda = x - \frac{2}{\sqrt{y}}$ and $\mu = -x - \frac{2}{\sqrt{y}}$ we find

$$u'_x = u'_\lambda - u'_\mu$$

$$u'_y = \frac{1}{y^{3/2}}(u'_\lambda + u'_\mu)$$

$$u''_{xx} = u''_{\lambda\lambda} - 2u''_{\lambda\mu} + u''_{\mu\mu}$$

$$u''_{yy} = -\frac{3}{2y^{5/2}}(u'_\lambda + u'_\mu) + \frac{1}{y^3}(u''_{\lambda\lambda} + 2u''_{\lambda\mu} + u''_{\mu\mu})$$

We find that

$$\begin{aligned} 0 &= 2yu''_{xx} - 2y^4 u''_{yy} - 3y^3\, u'_y \\ &= 2y(u''_{\lambda\lambda} - 2u''_{\lambda\mu} + u''_{\mu\mu}) - 2y^4\left(\frac{u''_{\lambda\lambda} + 2u''_{\lambda\mu} + u''_{\mu\mu}}{y^3} - \frac{3(u'_\lambda + u'_\mu)}{2y^{\frac{5}{2}}}\right) - \frac{3(u'_\lambda + u'_\mu)}{y^{3/2}} \end{aligned}$$

and, finally, $u''_{\lambda\mu} = 0$. The general solution to this equation is $u = f(\lambda) + g(\mu)$, the general solution to our equation is found as $u = F\left(x - \frac{2}{\sqrt{y}}\right) + G\left(x + \frac{2}{\sqrt{y}}\right)$, where F, G are arbitrary functions.

Problem 5(a). *Solve the initial value problem* $u''_{tt} - c^2 u''_{xx} = 0$, $x > 0, t > 0$, *subject to the initial conditions:* $u(0,t) = 1$ *for* $t > 0$; $u(x,0) = 1$, $u'_t(x,0) = \cos x - 1$ *for* $x > 0$.

Solution: method 1. Notice that a shifted function $v(x,t) = u(x,t) - 1$ is solution of

$$v''_{tt} - c^2 v''_{xx} = 0$$

with the initial conditions $v(0,t) = 0$ for $t > 0$; $u(x,0) \equiv g(x) = 0$, $u'_t(x,0) \equiv h(x) = \cos x - 1$ for $x > 0$. We have the exact formula for such an initial problem:

$$v(x,t) = \begin{cases} \frac{1}{2}[g(x+ct) + g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds, & \text{if } x \ge ct \ge 0 \\ \frac{1}{2}[g(x+ct) - g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds, & \text{if } 0 \le x \le ct \end{cases}$$

We find for $x \ge ct \ge 0$:

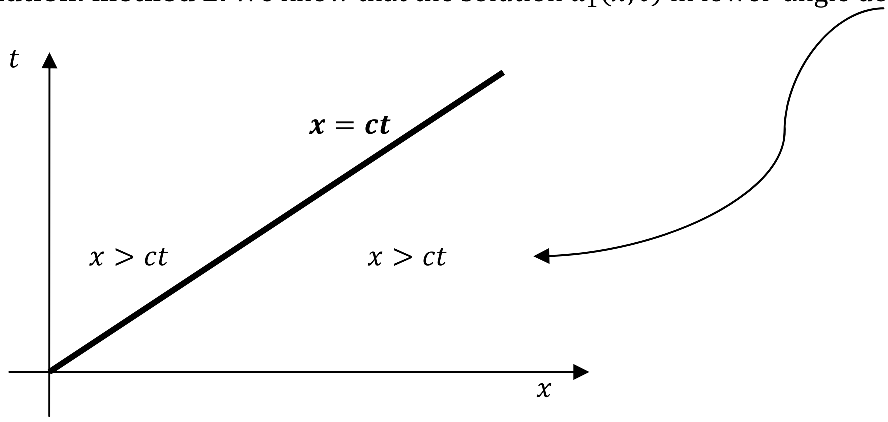
$$v(x,t) = \frac{1}{2}[g(x+ct) + g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds = \frac{1}{2c}\int_{x-ct}^{x+ct} (\cos s - 1)ds$$

hence $v(x,t) = \frac{1}{2c}(\sin(x+ct) - \sin(x-ct)) - t$. Since $g \equiv 0$ we conclude that the solution for $0 \le x \le ct$ has the same form. Hence, returning to u we obtain

$$u(x,t) = 1 + \frac{\sin(x+ct) - \sin(x-ct)}{2c} - t = 1 + \frac{1}{c}\sin ct \cos x - t$$

Problem 5(c). *Solve the initial value problem* $u''_{tt} - c^2 u''_{xx} = 0$, $x > 0, t > 0$, *subject to the initial conditions:* $u(0,t) = 1$ *for* $t > 0$; $u(x,0) = e^{-x^2}$, $u'_t(x,0) = 0$ *for* $x > 0$.

Solution: method 2. We know that the solution $u_1(x,t)$ in lower-angle domain $x > ct$



can be found as

$$u_1(x,t) = f(x+ct) + g(x-ct)$$

with the above initial conditions $u(x,0) = e^{-x^2}$, $u'_t(x,0) = 0$, $x > 0$. Hence,

$$u(x,0) = f(x) + g(x) = e^{-x^2} \quad \text{and} \quad u'_t(x,0) = c(f'(x) - g'(x)) = 0.$$

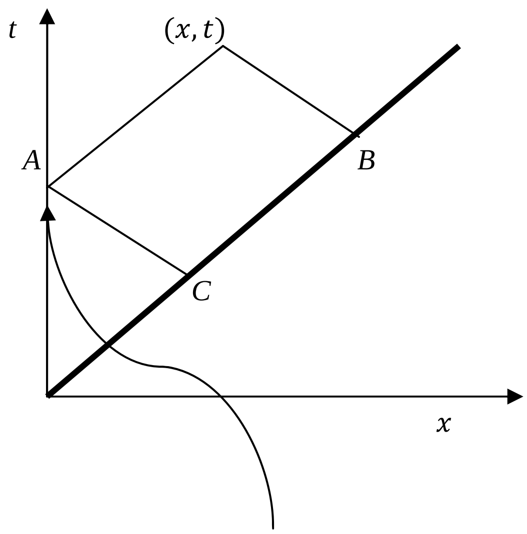
In other words, $g(x) = f(x) + C_1$. Therefore

$$f(x) + g(x) = 2f(x) + C_1 = e^{-x^2} \quad \Rightarrow \quad f(x) = \frac{1}{2}e^{-x^2} - \frac{C_1}{2},$$

which yields

$$u_1(x,t) = \left(\frac{1}{2}e^{-(x+ct)^2} - \frac{C_1}{2}\right) + \left(\frac{1}{2}e^{-(x-ct)^2} + \frac{C_1}{2}\right) = \frac{1}{2}\left(e^{-(x-ct)^2} + e^{-(x+ct)^2}\right)$$

In order to continue the found solution into the upper-angle domain, we apply the parallelogram rule (see picture below):



$$u_2(x,t) + u_2(C) = u_2(A) + u_2(B)$$

Here $u_2(A) = u(0, t_A) = 1$ by the initial conditions. On the other hand, $C = \left(\frac{ct-x}{2}, \frac{ct-x}{2c}\right)$ and $B = \left(\frac{ct+x}{2}, \frac{ct+x}{2c}\right)$, and we find (since $u_1 = u_2 = u$ on the "light line" $x = ct$)

$$u_2(x,t) = 1 + u_1\left(\frac{ct+x}{2}, \frac{ct+x}{2c}\right) - u_1\left(\frac{ct-x}{2}, \frac{ct-x}{2c}\right) = 1 + \frac{e^{-(x+ct)^2} - e^{-(x-ct)^2}}{2}$$

Combining the obtained solutions we van write

$$u(x,t) = \begin{cases} \frac{1}{2}\left(e^{-(x-ct)^2} + e^{-(x+ct)^2}\right), & 0 \le ct \le x \\ 1 + \frac{e^{-(x+ct)^2} - e^{-(x-ct)^2}}{2}, & 0 \le x \le ct \end{cases}$$

Remark: **Method 2** as in Problem 5(a) yields the same formulae (prove!).

Problem 8. *Let $\Omega = \{(x,y):\ 0 \le x \le \pi,\ 0 \le y \le \pi\}$. Solve the boundary problem*

$$u''_{xx} + u''_{yy} = 0, \qquad u(0,y) = u(\pi,y) = u(x,0) = 0, \qquad u(x,\pi) = 2\sin x - \sin 2x.$$

Solution. We apply the method by separating of variables. We know that the solution can be obtained by expanding in trigonometric series (see lecture notes 08)

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin nx \sinh ny$$

,

where the coefficients are found from the Fourier decomposition of $u(x,\pi) \equiv g(x)$:

$$g(x) = u(x,\pi) = \sum_{n=1}^{\infty} (B_n \sinh n\pi\,) \sin nx$$

In our case $g(x) = 2\sin x - \sin 2x$, hence comparing the coefficients we see that

$$B_1 \sinh \pi = 2, \qquad B_2 \sinh 2\pi = -1, \qquad B_n = 0 \text{ for } n \geq 3$$

We find $B_1 = \frac{2}{\sinh \pi}$ and $A_2 = -\frac{1}{\sinh 2\pi}$ which finally gives the solution

$$u(x,y) = \frac{2\sin x \sinh y}{\sinh \pi} - \frac{\sin 2x \sinh 2y}{\sinh 2\pi}$$

Problem 9. *Let $\Omega = \{(x,y):\ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ and u is a solution of $u''_{xx} + u''_{yy} = 0$. Solve the Dirichlet problem*

$$u(0,y) = 0, \qquad u(1,y) = y, \qquad u(x,0) = \sin 2\pi x\,, \qquad u(x,1) = \sin 3\pi x + x.$$

Solution. We make the following remarks: the function $u_1(x,y) = xy$ is *harmonic* and has boundary values:

$$u_1(0,y) = 0, \qquad u_1(1,y) = y, \qquad u_1(x,0) = 0, \qquad u_2(x,1) = x,$$

and similarly, the function $u_2(x,y) = \sin 2\pi x\ \cosh 2\pi y$ is harmonic with boundary values

$$u_2(0,y) = 0, \qquad u_1(1,y) = 0, \qquad u_1(x,0) = \sin 2\pi x\,, \qquad u_2(x,1) = \sin 2\pi x\ \cosh 2\pi\,,$$

Let $u(x,y)$ be the required solution. Then by linearity of the Laplace operator we have

$$v(x,y) = u(x,y) - u_1(x,y) - u_2(x,y) \qquad (*)$$

is a harmonic function with boundary conditions

$$v(0,y) = 0, \qquad v(1,y) = 0, \qquad v(x,0) = 0, \qquad v(x,1) = \sin 3\pi x - \sin 2\pi x\ \cosh 2\pi\,,$$

As in Problem 8 we find

$$v(x,y) = \sum_{n=1}^{\infty} B_n \sin n\pi x \sinh n\pi y$$

with the coefficients:

$$B_3 \sinh 3\pi = 1, \qquad B_2 \sinh 2\pi = -\ \cosh 2\pi\ , \qquad B_n = 0 \text{ for } n = 1, n \geq 4$$

We have

$$v(x,y) = \frac{\sin 3\pi x \sinh 3\pi y}{\sinh 3\pi} - \sin 2\pi x \sinh 2\pi y \frac{\cosh 2\pi}{\sinh 2\pi}$$

and now the required solution can be found from (*).

Problem 11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies*

$$\Delta u - u^3 = -u$$

in Ω and vanishes on the boundary: $u|_{\partial\Omega} = 0$. By using the maximum principle show that $-1 \leq u \leq 1$.

Solution. Denote by M and m the maximum and minimal values of our solution in $\bar{\Omega}$ (why do they exist?) Then we have to prove that $\max\{m, M\} \leq 1$. Let us argue by contradiction. That is we assume that $\max\{m, M\} > 1$. Since the equation and the zero-boundary condition do not change for change of function $u \to -u$, we can without less of generality to assume that, for instance, $M > 1$. Let x_0 be the point where the value M is achieved. Clearly $x_0 \in \Omega$, hence the function u is twice differentiable at x_0. We have

$$\Delta u(x_0) = -\big(1 - u(x_0)\big)u^2(x_0) = -(1 - M^2)M > 0.$$

But at the maximum point, the second differential is non-positively definite, that is the quadratic form

$$u''_{xx}(x_0)dx^2 + 2u''_{xy}(x_0)dxdy + u''_{yy}(x_0)dy^2 \leq 0.$$

In particular, it follows from standard linear algebra that $u''_{xx}(x_0) \leq 0,\ u''_{yy}(x_0) \leq 0$. But this contradicts $\Delta u(x_0) = u''_{xx}(x_0) + u''_{yy}(x_0) > 0$. The property is proved.