Problems, PDE, 2008-09-25

To pass test you need to solve at least 5 problems from the following list.

1. Find a solution to the Cauchy problem

$$(x+u+1)u'_x + 2u'_y = x+u-1, \qquad u(x,1)=0.$$

2. Prove that there is no real-valued solution of the equation $u'_x = \left(u'_y\right)^2$ in any open set which contains the straight line $y=0$ and which satisfies $u(x,0)=x^2$, $x\in\mathbb{R}$. Prove that there is a solution defined in an open set containing the straight line $y=0$ and which satisfies $u(x,0)=e^x$, $x\in\mathbb{R}$.

3. Solve the initial problem: $xu'_x + yu'_y + u'_x u'_y = 0,\ u(x,0)=x^2$.

4. Let $u(x,t)$ be the weak solution to Burgers equation $uu'_x + u'_t = 0$ with the initial data

$$u(x,0)=\begin{cases} 0, & x\le 0 \\ -x, & 0\le x\le 4 \\ -4, & x\ge 4 \end{cases}$$

a) By using the Rankine-Hugoniot condition find the equation of the shock line. b)Find an exact form of the weak solution.

5. By using Rankine-Hugoniot condition, find the weak solution to the Burger's equation

$$u'_t + uu'_x = 0$$

with the initial data

$$h(x)=\begin{cases} -1 & \text{if} \quad x<0 \\ -1-x & \text{if} \quad 0\le x\le 1 \\ -2 & \text{if} \quad x\ge 1 \end{cases}$$

6. Find the Legendre transform of the following functions:

(a) $L(p)=\sum_{k,j=1}^{n} c_{kj}q_k q_j$, where matrix (c_{kj}) is positively definite and symmetric;\
(b) $L(p)=\max\{|q_1|^2,|q_2|^2\}$

7. Obtain the solution to the following Cauchy problem

$$u'_t + \sum_{k=1}^{n}\left(u'_{x_k}\right)^2 = 0, \qquad u(x,0)=a|x|,$$

by using the Hopf-Lax formula.

8*. (Evans, Problem 8, p.163) Let E be a closed subset of $\mathbb{R}^n$. Show that if the Hopf-Lax formula could be applied to the initial-value problem

$$u'_t + |Du|^2 = 0,$$

with the initial data $u(x,0)=0$ when $x\in E$ and data $u(x,0)=+\infty$ when $x\notin E$ then it would give the solution $u(x,t)=\frac{1}{4t}\mathrm{dist}(x,E)^2$.