Problems, PDE, 2008-10-10

To pass test you need to solve at least one problem from the following list.

1. (Evans, p.290) Prove directly that if $u \in W^{1,p}(0,1)$ for some $1 < p < \infty$, then the
$$|u(x) - u(y)| \leq |x - y|^{1-\frac{1}{p}} \int_0^1 |u'(t)|^p dt$$

for a.e. $x, y \in [0,1]$.

2. Find an explicit description of the "primitive function" for distributions: prove that for each $F \in \mathcal{D}'(\mathbb{R}^1)$ there is a distribution $G \in \mathcal{D}'(\mathbb{R}^1)$ with $DG = F$, where $D = \frac{d}{dx}$.

3. Define the function $\psi: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ by
$$\psi = \begin{cases} \dfrac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} & t > 0 \\ 0 & t \leq 0 \end{cases}$$
   a) Show that ψ is locally integrable in $\mathbb{R}^2$, and, thus, defines a distribution $\tilde{\psi} \in \mathcal{D}'(\mathbb{R}^2)$
   b) Find $\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)(\tilde{\psi})$ in the weak sense.

4. By using co-area formula find a short derivation for the integral
$$\int_B \frac{dx}{|x|^n},$$

where B the ball
$$\{x \in \mathbb{R}^n\colon\ |x|^2 + a^2 < 2tx_1\}, \qquad 0 < a < t.$$

(*Hint*: use an auxiliary function $f(x) = \frac{|x|^2 + a^2}{2tx_1}$ and apply the mean value theorem for harmonic functions).