Motivations

A single conservation law in one space variable is a first order equation partial equation of the form

$$u_t' + \big(G(u)\big)_x' = 0 \tag{1}$$

- u is the conserved quantity
- $G(u)$ is the flux

Integration over some interval $[a, b]$ gives

$$\frac{d}{dt}\int_a^b u(t,x)dx = -\int_a^b G\big(u(t,x)\big)_x' \; dx$$

$$= G\big(u(a,t)\big) - G\big(u(b,t)\big) =$$

$$= [\text{inflow at } a] - [\text{outflow at } b]$$

In other words, the quantity u is never created nor destroyed: *the total amount of* u *contained inside of any given interval* $[a, b]$ *can change only due to flow of* u *across these two end-points*.

Another equivalent form of the above conservation law is

$$u_t' + a(u)u_x' = 0, \qquad a = G' \tag{2}$$

- For smooth solutions these two equations are equivalent
- If u has a jump, the second equation is not well-defined

However, the first equation (1) allows considering discontinuous solutions as well, interpreted in distributional sense. Namely, a locally integrable function $u = u(x,t)$ is a weak solution of (1) provided that

$$\iint_U (u\phi_t' + G(u)\phi_x')dxdt = 0 \tag{1w}$$

for every differentiable function ϕ with compact support $\phi \in C_0^1(U)$ for some specified domain $U \subset \mathbb{R}_+^2$.

In general, we consider the following system

$$\begin{cases} \dfrac{\partial}{\partial t}u_1 + \dfrac{\partial}{\partial x}G_1(u_1, \dots, u_n) = 0 \\ \qquad\qquad \dots \\ \dfrac{\partial}{\partial t}u_n + \dfrac{\partial}{\partial x}G_n(u_1, \dots, u_n) = 0 \end{cases} \tag{3}$$

These functions, unified in a vector $G = (G_1, \dots, G_n)$ can also be thought as a generalized flux.

How to get a system?

Example 1 (the p-system). A modified wave equation where the speed of wave propagation (or the flux) is a function of u'_x. Namely, we consider a nonlinear wave equation of the form

$$u''_{tt} - a(u_x)u''_{xx} = 0$$

(observe that for $a > 0$ we have a hyperbolic equation), where the flux depends on u'_x, then we can reduce this equation to a system if we set

$$w = u'_x, \qquad v = u'_t$$

Then

$$\begin{pmatrix} w'_t \\ v'_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a(w) & 0 \end{pmatrix} \cdot \begin{pmatrix} w'_x \\ v'_x \end{pmatrix}$$

This system takes the divergent form (1) if we set

$$a(w) = p'(w),$$

that is p is an anti-derivative of a. Then for the flux-function

$$G(w, v) = \left(-v, p(w)\right)^T$$

we obtain an equivalent description of our initial equation in the form (1).

Example 2 (Gas dynamics in Eulerian coordinates, "from the point of view of an observer")

$$\begin{cases} \dfrac{\partial}{\partial t}\rho + \dfrac{\partial}{\partial x}(\rho v) = 0 \\ \dfrac{\partial}{\partial t}(\rho v) + \dfrac{\partial}{\partial x}(\rho v^2 + p) = 0 \\ \dfrac{\partial}{\partial t}(\rho E) + \dfrac{\partial}{\partial x}(\rho E v + p v) = 0 \end{cases}$$

Here we see three conservations laws: for mass, momentum and energy respectively. An observer focuses on specific locations in the space through which the fluid flows: here v stands for velocity, ρ is density, p is pressure and E is energy. In order to define the solution one needs an additional equation, the so-called *equation of state*. For example, one can take an adiabatic relation

$$p(\rho) = C\rho^\gamma$$

Here γ is an adiabatic constant which for usual N-atomic gas is $\frac{2N+3}{2N+1}$; for instance $\gamma_{air} = \frac{7}{5} \sim 1.4$ for earth's atmosphere.

Definition. One says that the system (3) above is strictly hyperbolic if matrix

$$A(u) = DG(u)$$

has n real distinct eigenvalues, say

$$\lambda_1(u) < \cdots < \lambda_n(u).$$

Linear hyperbolic systems

Example 3. Sometimes solutions can be written explicitly. In the simplest case

$$u'_t + \lambda u'_x = 0, \qquad \lambda \in \mathbb{R}$$

we have homogeneous scalar Cauchy problem

$$u(x,0) = h(x)$$

and the solution can be found as follows:

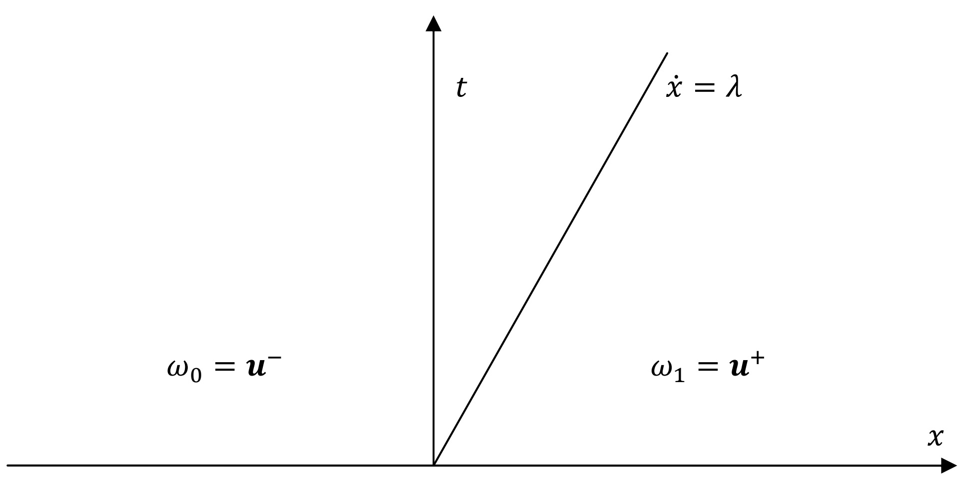
$$u(x,t) = h(x - \lambda t).$$

This function describes a travelling wave with the profile function h. Even h is non-differentiable, for instance, merely a L^1_{loc} function, the above solution can be interpreted as a weak solution in the distributional sense.

As a special case we consider the so-called Riemann initial data:

$$h(x) = \begin{cases} u^-, & if \;\; x < 0 \\ u^+, & if \;\; x > 0 \end{cases}$$

Then can be represented as follows:



Example 4. Similarly, one can solve an higher-dimensional analogue of the above equation

$$\boldsymbol{u}_t' + A\boldsymbol{u}_x' = 0, \qquad \boldsymbol{u}(x,0) = \boldsymbol{h}(x)$$

with A a *hyperbolic* matrix (a matrix with real eigenvalues). Let us look for a solution in the following form

$$\boldsymbol{u}(x,t) = \sum_{i=1}^{n} g_i(x - \lambda_i t)\boldsymbol{v}_i$$

where $\boldsymbol{g} = (g_1, \dots, g_n)$ are some unknown vector-function and $\boldsymbol{v}_j$ are eigenvectors associated with λ_i. Such an a priori given solution is what is usually called an *ansatz* (an assumed form of solution). Then the substitution of the ansatz gives us

$$\boldsymbol{u}_t' + A\boldsymbol{u}_x' = -\sum_{i=1}^{n} g_i'(x - \lambda_i t)\lambda_i \boldsymbol{v}_i + \sum_{i=1}^{n} g_i'(x - \lambda_i t)A\boldsymbol{v}_i = 0$$

Thus we need only to find functions g_i to be consistent with the initial conditions. We have the following linear equation

$$\boldsymbol{u}(x,0) \equiv \sum_{i=1}^{n} g_i(x)\boldsymbol{v}_i = \boldsymbol{h}(x)$$

which obviously has a unique solution since our eigenvectors form a basis for any x.

Thus, we have a "wave packet", that is a solution is a sum of simple waves.

Consider again the Riemann initial data:

$$\boldsymbol{h}(x) = \begin{cases} \boldsymbol{u}^-, & if \quad x < 0 \\ \boldsymbol{u}^+, & if \ \ x > 0 \end{cases}$$

Now the corresponding solution can be obtained as follows. Write the vector $\boldsymbol{u}^+ - \boldsymbol{u}^-$ as a linear combination of eigenvectors of A, i.e.

$$\boldsymbol{u}^+ - \boldsymbol{u}^- = \sum_{i=1}^{n} c_i(x)\boldsymbol{v}_i$$

And define the intermediate states

$$\omega_j = \boldsymbol{u}^- + \sum_{i=1}^{j} c_i(x)\boldsymbol{v}_i\,, \qquad \omega_0 \coloneqq \boldsymbol{u}^-, \qquad \omega_n = \boldsymbol{u}^+$$

So that each difference

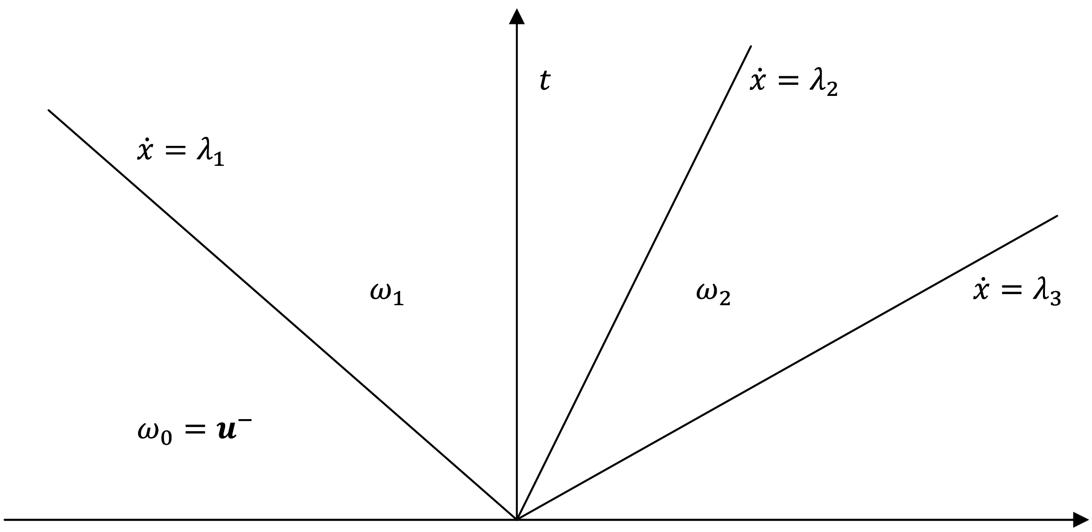
$$\omega_j - \omega_{j-1} = c_j(x)\boldsymbol{v}_j$$

is a j-eigenvector of A.

Then solution takes the form

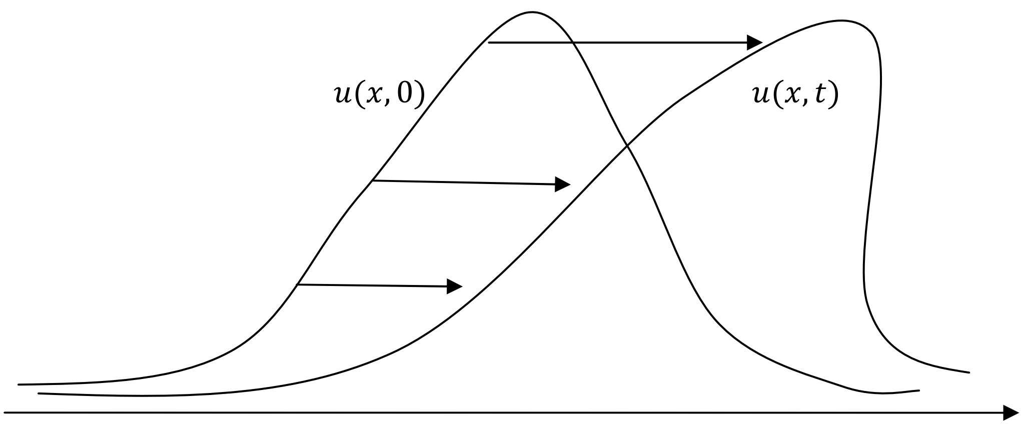
$$u(x,t) = \omega_j, \qquad if\ \lambda_j < \frac{x}{t} < \lambda_{j+1}$$

(for convenience we set $\lambda_0 = -\infty$ and $\lambda_{n+1} = +\infty$). See the picture below.



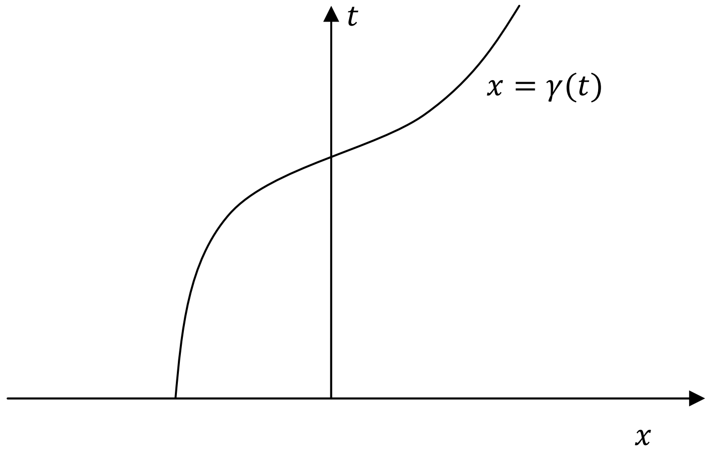
Loss of regularity

A basic feature of nonlinear systems of the form is that, even for smooth initial data, the solution of the Cauchy problem may develop discontinuities in finite time.



The solution can be prolonged for supercritical times only within the class of discontinuous function.

Similarly to the one-dimensional case, one can derive the conditions imposed by the weak problem above (1w) on a solution at points of jump. We assume that $u(x,t)$ has a discontinuity across a line $x = \gamma(t)$.



Call

$$u^{\pm}(t) = \lim_{x \to \gamma(t)\pm} u(x,t)$$

the right and left limits of $u(x,t)$ at the point of jump. Applying the divergence theorem to (1w) (more precisely, to the vector-field $(u\phi, G(u)\phi)$) we obtain for any differentiable function ϕ with compact support

$$\iint (u\phi_t' + G(u)\phi_x')dxdt = -\iint (u_t' + A(u)u_x')\phi\, dxdt\ +$$

$$+\int \left\{\left(u^+(t) - u^-(t)\right)\dot{\gamma}(t) - \left(G\left(u^+(t)\right) - G\left(u^-(t)\right)\right)\right\}\phi\left(t,\gamma(t)\right) dt$$

Thus we obtain the general Rankine-Hugoniot condition (in vector form!)

$$(u^+ - u^-)\dot{\gamma} = G(u^+) - G(u^-) \tag{HR}$$

If we define the averaged matrix

$$A(u,v) := \int_0^1 A(\theta u + (1-\theta)v)\, d\theta,$$

where $A = DG$ is the Jacobi matrix. Then the above condition can be written in the following equivalent form

$$\dot{\gamma}(u^+ - u^-) = G(u^+) - G(u^-) = \int_0^1 \frac{d}{d\theta} G(u^+\theta + (1-\theta)u^-)\, d\theta = A(u^+, u^-)\cdot(u^+ - u^-)$$

It follows that $(u^+ - u^-)$ is an eigenvector of the averaged matrix $A(u^+, u^-)$ and $\dot{\gamma}$ coincides with the corresponding eigenvalue.

Remark. In the scalar case we have

$$\dot{\gamma} = \frac{G(u^+) - G(u^-)}{u^+ - u^-} = \frac{1}{u^+ - u^-}\int_{u^-}^{u^+} G'(\theta)\, d\theta$$

One usually requires in the one-dimensional case that G is a convex function.

Admissibility conditions

In the presence of discontinuities, the Rankine-Hugoniot equations may not suffice to single out a unique solution to the Cauchy problem.

For example, if we consider the scalar Burger's equation with $G(u) = \frac{u^2}{2}$ and with initial data

$$u(x,0) \equiv h(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

then we obtain infinitely many solutions. Namely, for any $a \in [0,1]$ the piecewise constant function

$$u(x,t) = \begin{cases} 0 & x \leq \frac{at}{2} \\ a & \frac{at}{2} < x < \frac{(1+a)t}{2} \\ 1 & x \geq \frac{(1+a)t}{2} \end{cases}$$

provides a solution in distributional sense. Indeed, the Rankine-Hugoniot conditions hold along the two lines of discontinuity

$$\xi_1(t) = \frac{at}{2}, \qquad \xi_2(t) = \frac{(1+a)t}{2}.$$

From the previous example it is clear that, in order to achieve a theorem stating uniqueness and continuous dependence on the initial data, the notion of weak solution must be supplemented with further "admissibility conditions", possibly motivated by physical considerations. We discuss the most well-known condition which follows form approximation of our 1st order hyperbolic system by a perturbed 2nd order system with a viscous term.

Vanishing viscosity

A weak solution u is admissible if there exists a sequence of smooth solutions $u^\epsilon(x,t)$ to

$$\frac{\partial}{\partial t} u^\epsilon(x,t) + A\big(u^\epsilon(x,t)\big)\frac{\partial}{\partial x} u^\epsilon(x,t) = \epsilon \frac{\partial^2}{\partial x^2} u^\epsilon(x,t)$$

which converges in L^1 as $\epsilon \to 0$.

A C^1-smooth function $\eta: \mathbb{R}^n \to \mathbb{R}^1$ is called an entropy for the system (1) with entropy flux $q: \mathbb{R}^n \to \mathbb{R}^1$ if

$$D\eta(\xi) \cdot DG(\xi) = Dq(\xi), \qquad \forall\, u \in \mathbb{R}^n$$

Notice that the entropy relation above yields that if $u(x,t)$ is some C^1-smooth solution of (1) then

$$\left(\eta(u)\right)'_t + \left(q(u)\right)'_x = 0$$

Indeed,

$$D\eta(u)u'_t + Dq(u)\,u'_x = D\eta(u)\,(-DG(u)\,u'_x) + Dq(u)\,u'_x = 0$$

Hence we have an additional conservation law.

Assume that the entropy η is a convex and C^2-smooth.

Definition. A weak solution $u(x,t)$ of (1) is entropy admissible if

$$\left(\eta(u)\right)'_t + \left(q(u)\right)'_x \leq 0$$

in the sense of distributions, that is

$$\iint (\eta(u)\phi'_t + q(u)\phi'_x)dxdt \leq 0,$$

for any pair (η, q), where η is a convex entropy and q is the corresponding entropy flux.

Remark. In the scalar case one can consider

$$\eta(u) = |u - c|, \qquad q(u) = \mathrm{sgn}\,(u - c) \cdot \left(f(u) - f(c)\right)$$

for any real $c \in \mathbb{R}$.