Partial Differential Equations / Partiella differentialekvationer

- **1TT462 (6.0 hp)**
- **1MA054 (5.0 hp)**

Lecturer: *Vladimir Tkachev*, guest professor, KTH

Webb: http://www.math.kth.se/~tkatchev/teaching/index.html

Email: tkatchev@kth.se

Goals

The course aims at developing the theory for hyperbolic, parabolic, and elliptic partial differential equations in connection with physical problems.

Contents

- Characteristics
- 1st order PDE
- Linear second order PDE: the Laplace and Poisson equations, the wave equation and the heat equation
- Sobolev spaces
- Linear elliptic equations
- Systems of conservations laws

Examinationsform (4 poäng)

Skriftligt prov med problem och teoriuppgifter vid kursens slut.

Examinationsform (6 poäng)

Ett skriftligt och i allmänhet ett muntligt prov ges vid kursens slut.

Dessutom förekommer obligatoriska inlämningsuppgifter eller ett teoriprov som redovisas i skriftlig och/eller muntlig form.

Deltagarna förväntas utföra ett projektarbete.

Course material:

- Lawrence C. Evans. *Partial Differential Equations*
- Robert C. McOwen, *Partial Differential Equations, Methods and Applications*

Plan

- Introduction, course information

- Review of ordinary differential equations (existence, uniqueness and non-uniqueness)
    - Existence of solutions
    - Exact solutions
        - separable equations
        - homogeneous equations
        - linear equations (higher orders)

- Definition of PDE
    - PDE are (much) more difficult than ODE
    - Systems of ODE ⟷ 1^{st} order PDE

- Why does one study PDE?
    - Example of derivation of PDE

- Classes of PDE
    - Variational problems (conservation laws)
    - Kinetics, gas dynamics (evolution)
    - Free-boundary problems/phase transitions
    - Solitons and exactly solvable (*integrable*) eqns.
    - etc…

- 1^{st} order PDE
    - Characteristic curves for a linear equation
    - Why Cauchy problem?

Some important examples

- Inviscid Burger's (Hopf) equation

$$u'_t + uu'_x = 0$$

- Scalar conservation law

$$u'_t + \operatorname{div} \mathbf{F}(u) = 0$$

- Laplace Equation

$$\Delta u \equiv u''_{xx} + u''_{yy} = 0$$

- Poisson's equation

$$\Delta u \equiv u''_{xx} + u''_{yy} = f(x, y)$$

- Heat (or diffusion) equation

$$u'_t - \Delta u = 0$$

- Wave equation

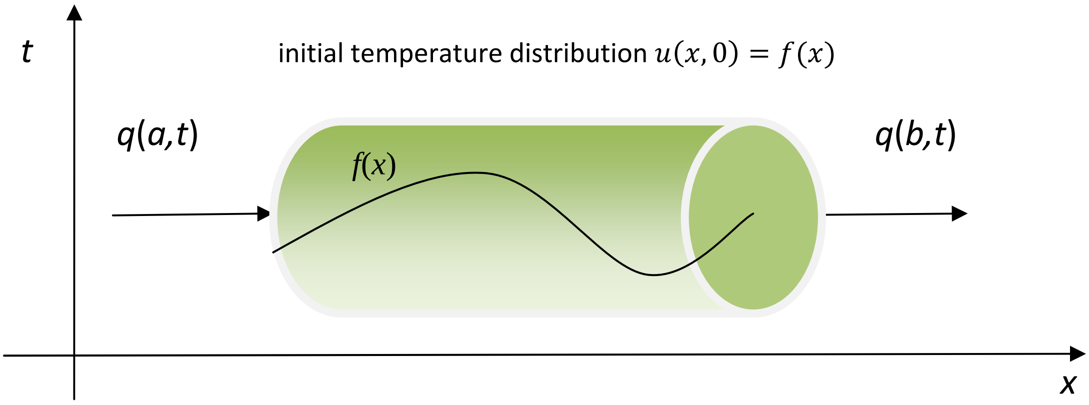
$$u''_{tt} - \Delta u = 0$$

- Minimal surface equation

$$\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0$$

Derivation of the heat equation

We consider the flow of heat along a metal *insolated* rod



Energy of an arbitrary piece of rod from a to b is

$$E = \int_a^b A \cdot \rho\, c \cdot u(x,t)\, dx$$

- $u = u(x,t)$ is temperature at time *t* at a given point x,
- A is the cross sectional area of the rod,
- c is the specific heat capacity of the rod

The wave heat flow:

$$R = A\big(q(a,t) - q(b,t)\big) = -A \int_a^b \frac{\partial}{\partial x} q(x,t)\, dx$$

Conservation of energy (in terms of power = time-derivative of energy):

$$R = \frac{\partial E}{\partial t}$$

implies the *integral form* of the heat equation:

$$\int_a^b \left(\rho\, c \cdot \frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} q(x,t)\right) dx = 0.$$

By virtue of arbitrariness of a and b we get

$$\rho\, c \cdot \frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} q(x,t) = 0.$$

Finally, by using the Fourier law $q(x,t) = -\lambda \frac{\partial}{\partial x} q(x,t)$ we arrive at (the differential form of) the heat equation:

$$\frac{\partial u}{\partial t} - C\, \frac{\partial^2 u}{\partial x^2} = 0.$$