**Classification of the 1st order PDE's**

**Standard notation:**

For $u = u(x,y)$ one denotes the first derivatives by

$p = u'_x$

$q = u'_y$

- The most general 1st order PDE

$$F(x,y,u,p,q) = 0$$

- A **linear** equation

$$a(x,y)u'_x + b(x,y)u'_y = c(x,y)$$

- A **homogeneous** (linear) equation

$$a(x,y)u'_x + b(x,y)u'_y = 0$$

Generalizations of the linear case:

- A **semilinear** equation

$$a(x,y)u'_x + b(x,y)u'_y = c(x,y,u)$$

- A **quasilinear** equation

$$a(x,y,u)u'_x + b(x,y,u)u'_y = c(x,y,u)$$

**Fully non-linear** equation:

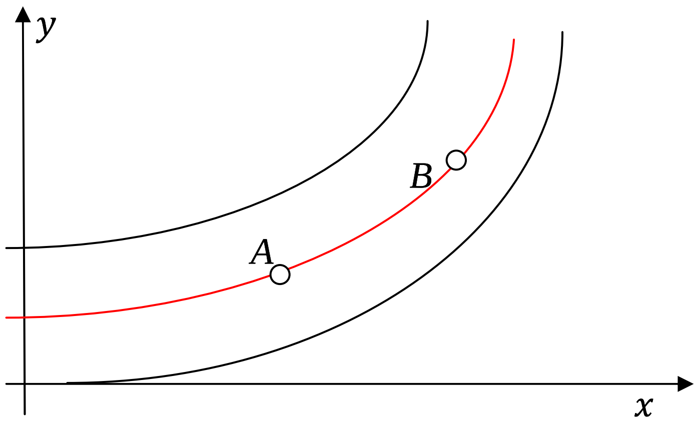
$$F(x,y,u,p,q) = 0$$

with *F* chosen arbitrarily; then additionally required that

$$F_p'^2 + F_q'^2 \neq 0$$

1. Characteristics for a homogeneous linear equation

$$a(x,y)u'_x + b(x,y)u'_y = 0$$



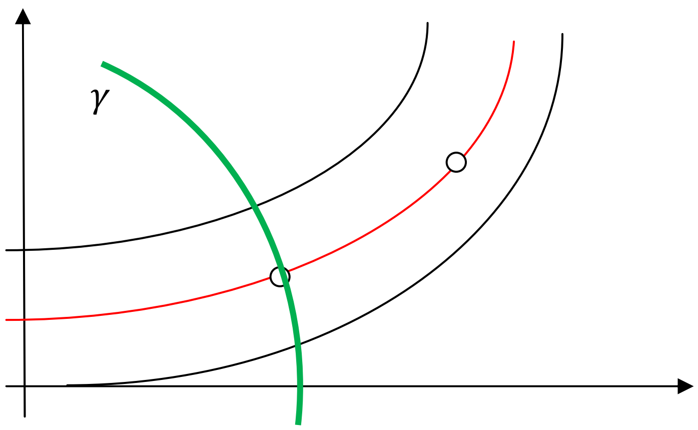

Characteristic equations:

$$\frac{dx}{dt} = a(x,y), \qquad \frac{dy}{dt} = b(x,y)$$

this yields

$$\frac{du(x(t),y(t))}{dt} = 0$$

hence $u(x,y) = \text{const}$ along each characteristic curve. In particular $u(A) = u(B)$ on the picture above and one can determine solution (uniquely) if one knows the values of the solution at some points. For instance, if one knows the values along a curve $\gamma$ which is transversal to characteristic curves:



2. The method of characteristics for general quasilinear equation

$$a(x,y,u)u'_x + b(x,y,u)u'_y = c(x,y,u)$$

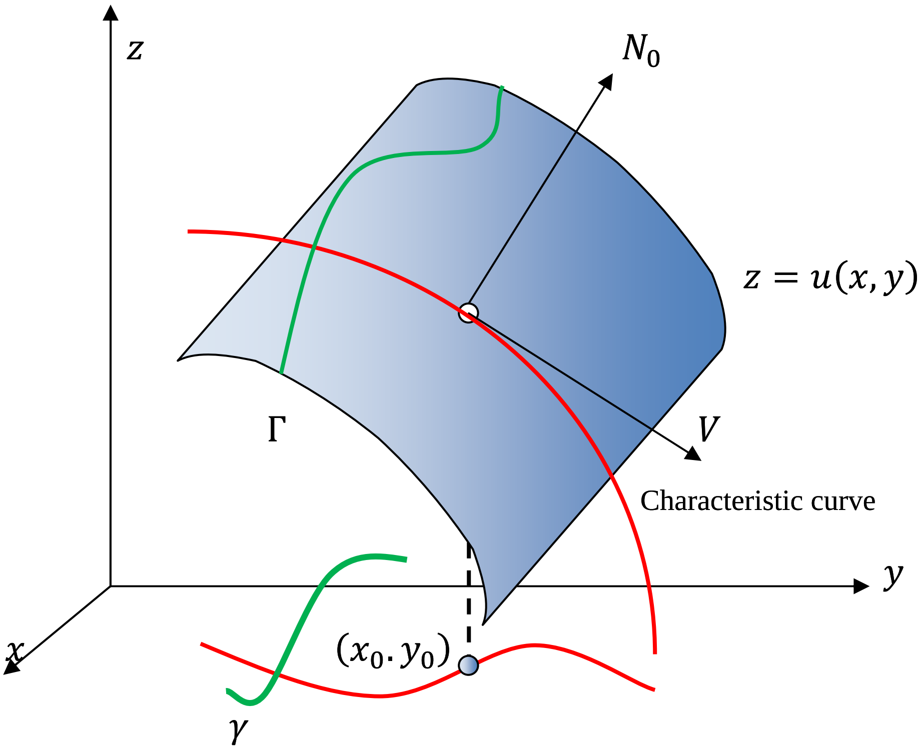
If we introduce a vector field $V = (V_1, V_2, V_3)$ with coordinates

$$V_1 = a(x,y,u), \qquad V_2 = b(x,y,u), \qquad V_3 = c(x,y,u)$$

then $V$ is orthogonal to the normal vector

$$N_0 = (-u'_x(x_0.y_0), -u'_y(x_0.y_0), 1)$$

at the point $(x_0, y_0, u(x_0, y_0))$ on the graph of a solution $z = u(x,y)$:




- $z = u(x,y)$ are *integral surfaces* of the vector field $V$
- a characteristic curve (in red)
- a Cauchy data $\Gamma$ (in green): a curve in $\mathbb{R}^3$ transversal to the vector field $V$

Characteristic equations:

$$\frac{dx}{dt} = a(x,y,u), \qquad \frac{dy}{dt} = b(x,y,u), \qquad \frac{dz}{dt} = c(x,y,u)$$

**The Cauchy problem**: given a curve $\Gamma$ in $\mathbb{R}^3$, find a solution $u$ of the 1st order equation whose graph contains $\Gamma$:

$$u|_\gamma = h(x,y)\ .$$

**The inviscid Burgers' equation**

$$u\,u'_x + u'_y = 0$$

or equivalently

$$\left(\frac{u^2}{2}\right)'_x + u'_y = 0$$

is an example of a conservation law in the general form

$$(G(u))'_x + u'_y = 0$$

*Mechanical interpretation:* 1D stream of particles is in motion, each particle having constant velocity; a velocity field is given by $u(x,y)$, where $y$ denotes time. If we follow an individual particle, we get a function $x = x(t)$ for which $u(x(t),t)$ remains constant:

$$0 = \frac{d}{dt}\big(u(x(t),t)\big) = u'_x(x(t),t)\cdot\frac{dx(t)}{dt} + u'_y(x(t),t)\cdot\frac{dt}{dt} = u\,u'_x + u'_y$$

An initial velocity (the Cauchy problem):

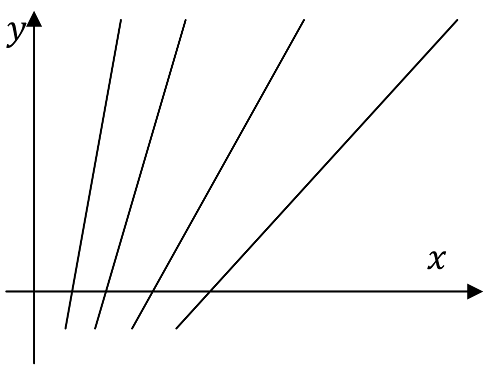
$$u(x,0) = h(x).$$

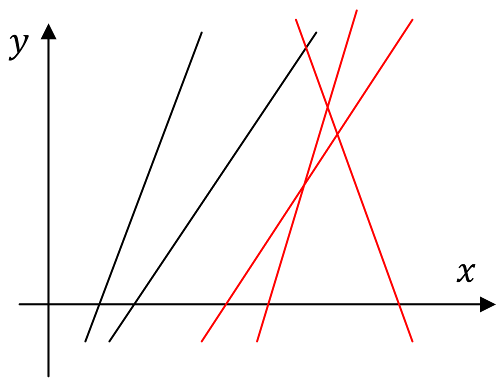
- Characteristics lines are

$$x = h(s)t + s, \qquad y = t, \qquad z = h(s) \qquad (**)$$

- Then the general solution is $u = h(x - uy)$

One must distinguish two cases subject to the global behavior of the solution:




(I) $h'(s) \geq 0$ for all $s$ (II) $h'(s_0) < 0$ for some $s_0$

Indeed, a simple analysis of (**) shows that two characteristics will intersect if and only if the system

$$x = h(s_1)t + s_1$$

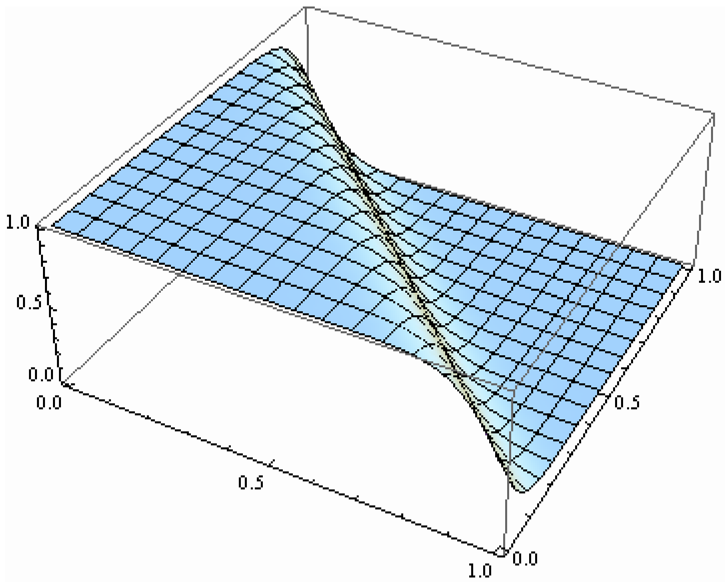
$$x = h(s_2)t + s_2$$

has a *positive* solution *t* (i.e. for positive time), which is equivalent to saying that

$$t = \frac{s_2 - s_1}{h(s_1) - h(s_2)} > 0$$

If this condition is satisfied for some values $s_1$ and $s_2$then solution suffers a *gradient catastrophe* type of singularity. Otherwise the solution exists globally.

**An example** of the gradient catastrophe**:**



**Weak solutions for quasilinear equations**

Demonstration by an example of a conservation law in the form

$$(G(u))'_x + u'_y = 0 \qquad (*)$$

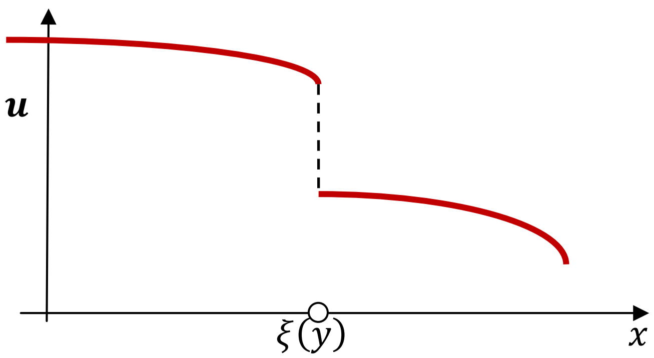
where $G$ is some smooth function of $u$.

The weak form of (*), also called a conservation law, is the following integrated identity (cf. the integral form of the heat equation given in lecture 1)

$$G\big(u(b,y)\big) - G\big(u(a,y)\big) + \frac{d}{dy}\int_a^b u(x,y)\,dx = 0.$$

Roughly speaking, a weak solution may contain discontinuities, may not be differentiable, and will require less smoothness to be considered a solution than a classical solution. Working with the weak solution of a PDE usually requires that the PDE be reformulated in an integral form. If a classical solution to the problem exists, it will also satisfy the definition of a weak solution.

We consider the simplest case when discontinuity of $u$, also called a shock front, is projected to a smooth curve front $x = \xi(y)$ in the $xy$-plane. In other words, for a fixed $y$ function $u$ has a jump discontinuity at $x = \xi(y)$:




Then we have the following necessary condition for the shock front $x = \xi(y)$:

**Jump condition (or the Rankine-Hugoniot condition):**

$$\xi'(y) = \frac{G(u_r) - G(u_l)}{u_r - u_l}$$

Here $u_l$ and $u_r$ denote the limiting values of $u$ from the left and right sides of the shock.